M_{Pl} barely depends on size L of the extra dimension! That is drastically different from the case of flat extra dim.!

Thus M_{Pl} in 4d remains large $\sim M_*$, and therefore gravity is weak!
→ The 4d hierarchy problem is resolved in this model!

3) Radius stabilization in RS1
(Goldberger - Wise mechanism)

Problem is that in RS1 as discussed so far there is no potential for the size-parameter L, implying the existence of a massless scalar field (radion).
→ Why is the size $k \cdot L \sim 35$ natural, and how could it be stable?

In the Goldberger - Wise mechanism this problem is fixed by introducing a massive scalar field in the bulk in a suitable way.

To obtain a potential for L with a minimum one needs two "forces" pulling in opposite directions (like for stable orbits in a two-body problem).
Here the mass of the scalar field will tend to make the size smaller.
In addition, one introduces a nontrivial profile of the scalar field VEV along the y-direction which tends to make the size larger (in order to have a smaller slope).
Technically, the latter is done by introducing brane potentials which fix the VEV at 0 and L at two different values.
The corresponding actions are

$$S^{\phi}_{bulk} = \frac{1}{2}\int d^4x \int_{-L}^{L} \sqrt{G}\,\left(G^{AB}\partial_A\phi\,\partial_B\phi - m^2\phi^2\right)$$

$$S^{\phi}_{Pl} = -\int d^4x\,\sqrt{g_{Pl}}\,\lambda_{Pl}\left(\phi^2 - v_{Pl}^2\right)^2$$

$$S^{\phi}_{TeV} = -\int d^4x\,\sqrt{g_{TeV}}\,\lambda_{TeV}\left(\phi^2 - v_{TeV}^2\right)^2$$

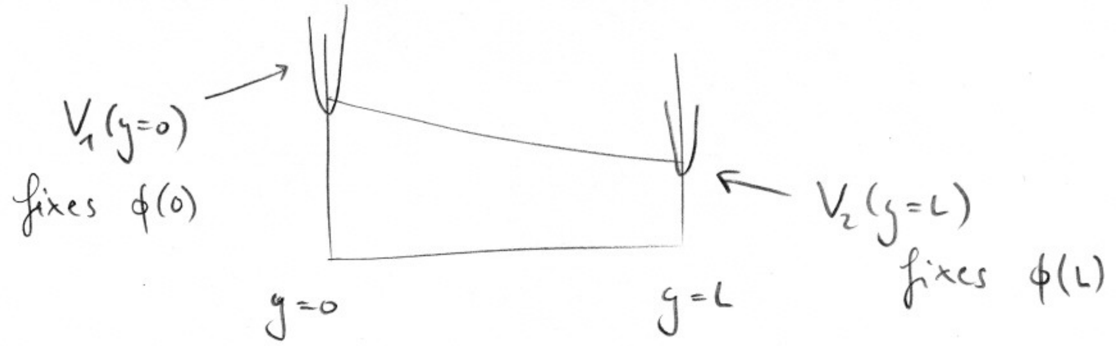
The general solution in the bulk is found as

$$\phi(y) = e^{2k|y|}\left(A\, e^{\nu k|y|} + B\, e^{-\nu k|y|}\right)$$

with

$$\nu = \sqrt{4 + \frac{m^2}{k^2}}$$

The brane potentials are chosen such that the situation looks like



Then one can write a potential for L, $V(L)$, and minimize it. One finds for the minimum

$$k\cdot L = \frac{4}{\pi}\,\frac{k^2}{m^2}\,\ln\left(\frac{v_{Pl}}{v_{TeV}}\right)$$

For $k\cdot L \simeq 35$ one needs $k > m$, but not by much, which is a simple requirement without fine-tuning.

Note that due to the potential for L (that is due to stabilization) the radion acquires a mass. $\rightarrow$ important for radion phenomenology!

The Goldberger - Wise mechanism was later improved by including the backreaction of the ϕ - energy density on the space - time curvature.

4) Gravity in RS 1

Now study KK - decomposition of graviton in AdS background.

Usual expectation from flat extra dim. on S^1 is:

zero modes: $\begin{cases} \text{graviton} & \\ \text{vector} & (\text{"graviphoton"}) \\ \text{scalar} & (\text{"graviscalar"}) \end{cases}$

+ massive graviton at higher KK levels (having eaten scalar + vector)

But here situation is different due to orbifold! Consider generic form of metric with fluctuations:

$$ds^2 = e^{-2k|y|} \underbrace{g_{\mu\nu}}_{\substack{\text{contains graviton} \\ \text{(see below)}}} dx^\mu dx^\nu + \underbrace{A_\mu}_{\substack{\text{vector} \\ \text{fluctuation}}} dx^\mu dy - \underbrace{b^2}_{\substack{\text{scalar} \\ \text{fluctuation}}} dy^2$$

Since ds^2 is symmetric under $y \leftrightarrow -y$, A_μ has to change sign under that transf. and hence cannot have a zero-mode. Therefore there will only be a scalar and the graviton zero-mode, plus massive gravitons at higher KK levels.

We first concentrate only on the graviton and set b to zero.

To find KK expansion of the graviton go to <u>conformal frame</u>,

$$ds^2 = e^{-A(z)} \left[\left(\eta_{\mu\nu} + h_{\mu\nu}(x,z)\right) dx^\mu dx^\nu - dz^2 \right]$$

with

$$e^{-A(z)} = \frac{1}{(1+k|z|)^2}$$

or

$$A(z) = 2 \log (k|z|+1) .$$

The relation between y and z is

$$\frac{1}{1+k|z|} = e^{-k|y|}$$

We look for linearized fluctuations around the background satisfying ("perturbed") Einstein equation,

$$\delta G_{MN} = \frac{1}{M_*^3} \delta T_{MN} .$$

We fix the so-called RS - gauge

$$h^\mu{}_\mu = \partial_\mu h^\mu{}_\nu = 0 ,$$

and expand δG_{MN} keeping only linear terms in order to get linearized Einstein eq. in warped background,

$$-\frac{1}{2} \partial^R \partial_R h_{\mu\nu} + \frac{3}{4} \partial^R A \ \partial_R h_{\mu\nu} = 0 .$$

It is convenient to use the rescaling

$$h_{\mu\nu} = e^{\frac{3}{4}A} \tilde{h}_{\mu\nu} ,$$

giving

$$-\frac{1}{2} \partial^R \partial_R \tilde{h}_{\mu\nu} + \left[\frac{9}{32} \partial^R A \ \partial_R A - \frac{3}{8} \partial^R \partial_R A \right] \tilde{h}_{\mu\nu} = 0$$

which has the form of a one-dimensional Schrödinger equation.

(Note $\partial^R \partial_R = -\Box_x - \nabla_z^2$.)

Now separate variables

$$\tilde{h}_{\mu\nu}(x,z) = \hat{h}_{\mu\nu}(x)\, \psi(z)$$

and require $\hat{h}_{\mu\nu}$ to be a 4d mass eigenstate with

$$\Box \hat{h}_{\mu\nu} = m^2 \hat{h}_{\mu\nu} .$$

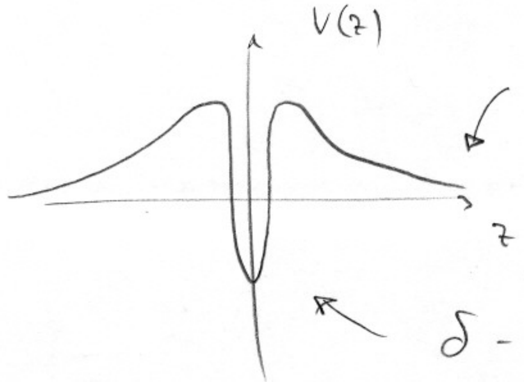
Then the Schrödinger equation for the KK modes becomes

$$-\partial_z^2 \psi + \underbrace{\left(\frac{9}{16} A'^2 - \frac{3}{4} A''\right)}_{= V(z) \text{ Schrödinger potential}} \psi = m^2 \psi$$

We have $A = 2 \log (k|z| + 1)$, so

$$V(z) = \frac{15}{4} \frac{k^2}{(1+k|z|)^2} - \frac{3k}{1+k|z|} \delta(z) ,$$

the so-called <u>volcano potential</u>



fall-off $\sim \frac{1}{z^2}$ at large z

δ - fct. at $z = 0$

As known from quantum mechanics the δ-fct potential has a single bound state, here: the massless graviton (which was expected since 4d Lorentz invariance is unbroken).

Explicitly, one finds solution

$$\psi^{(0)}(z) = e^{-\frac{3}{4}A(z)}$$

$$= \frac{1}{(1+k|z|)^{3/4}}$$

or in y-coordinates with

$$ds^2 = e^{-2k|y|}(\eta_{\mu\nu} + h_{\mu\nu}) - dy^2$$

(confirming the ansatz made earlier) one has

$$\psi^{(0)}(y) = e^{-\frac{3}{4}k|y|} .$$

This means that the graviton zero-mode is <u>localized at</u> the <u>Planck brane</u> ($y=0$) but exponentially suppressed ($\sim e^{-\frac{3}{4}kL}$, $kL \simeq 35$) at the TeV brane, without introducing small parameters.

→ <u>Gravity</u> is <u>localized</u> around the positive tension brane.

Here the weakness of gravity at the TeV-brane is due to the lokalization of the graviton wave function at the other brane!

(Note that this mechanism is completely different from flat (large) extra dimensions!)

Higher modes, on the other hand, are pushed to large $|z|$ by the barrier in the volcano potential and cannot easily get to the Planck brane (only by tunnelling → estimate possible with WKB methods).

To find wave functions of higher KK modes impose suitable boundary conditions obtained from orbifold conditions under $y \to -y$. For graviton that implies

$$\partial_y h_{\mu\nu} = 0 \quad \text{at } y = 0 \text{ and } L,$$

or $$\partial_z h_{\mu\nu} = 0 \quad \text{in } z\text{-coord. at branes.}$$

That gives

$$\partial_z f = -\frac{3}{2} k f \Big|_{z = z_{Pl} = 0}$$

$$\partial_z f = -\frac{3}{2} \frac{k}{1+k|z|} f \Big|_{z = z_{TeV} = \frac{1}{k} e^{kL}}$$

Then in the bulk

$$-\partial_z^2 f + \frac{15}{4} \frac{k^2}{(1+k|z|)^2} f = m^2 f ,$$

with general solution

$$f(z) = \frac{1}{\sqrt{1+k|z|}} \left[a_m Y_2\left(\frac{m(z+1)}{k}\right) + b_m J_2\left(\frac{m(z+1)}{k}\right)\right]$$

where Y_2 and J_2 are Bessel functions and a_m and b_m are determined by boundary conditions.

The mass spectrum found from this is

$$m_j = x_j \, k \, e^{-kL}$$

with x_j the roots of the Bessel function J_1, $J_1(x_j) = 0$. Approximately, they are

j	x_j
1	3.8
2	7.0
3	10.2
4	16.5
⋮	⋮

Note that the modes are <u>not</u> evenly spaced. Since $k e^{-kL} \sim O(\mathrm{TeV})$, the massive gravitons have masses of $O(\mathrm{TeV})$.

Due to the barrier in the volcano potential these <u>massive gravitons</u> have wave functions <u>peaked near</u> the <u>TeV-brane</u>. Therefore their <u>coupling</u> to TeV-brane matter is <u>exponentially</u> <u>enhanced</u>.

[This can also be seen from

$$\frac{\psi(z_{TeV})}{\psi(z_{Pl})} \sim e^{kL}$$

$\uparrow$ <u>inverse</u> warp factor]

Therefore the interaction of TeV-brane matter with the graviton zero-mode and with the massive graviton modes is very different:

$$\mathcal{L}^{\text{grav-matter}}_{TeV} = -\frac{1}{M_{Pl}} T^{\alpha\beta} h^{(0)}_{\alpha\beta}$$

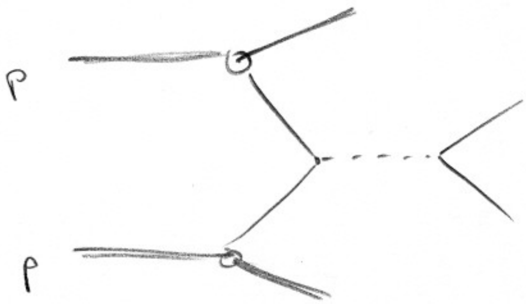
$$-\frac{1}{\underbrace{M_{Pl} e^{-kL}}_{O(\mathrm{TeV})}} T^{\alpha\beta} \sum_{n=1}^{\infty} h^{(n)}_{\alpha\beta}$$

5) Phenomenology of RS 1

The massive gravitons couple strongly to matter on the TeV-brane. As we have seen, this holds even for individual modes. Therefore they decay quickly.

Due to the discrete spectrum with spacings $\Delta m \sim \mathcal{O}(\mathrm{TeV})$ we can expect to see single resonances.

In hadron-hadron collisions possible channels will be for example



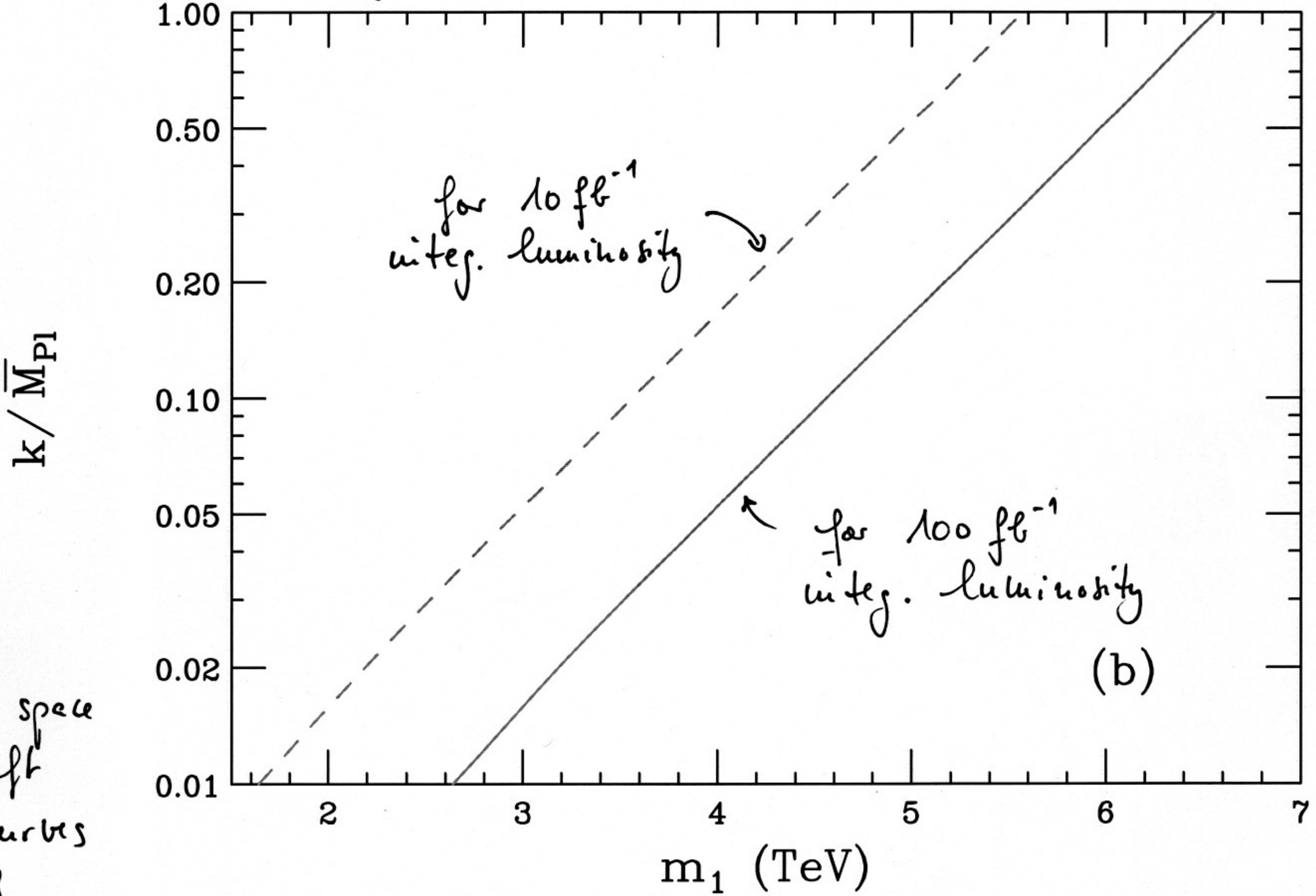
$$gg,\ q\bar{q} \to G^{(1)} \to l^+ l^-$$

$$gg,\ q\bar{q} \to G^{(1)} \to \bar{q}q,\ gg$$

corresponding to lepton pairs or 2-jet events with resonances in the invariant mass spectrum.

With these events one can test a large region in the parameter space (warped Planck scale / AdS curvature) at LHC, see figure.

LHC reach for discovery / exclusion of first massive KK graviton in RS1



mass of first KK resonance

parameter space
to the left
of the curves
is covered

An even cleaner signal can be expected at a future e^+e^- - Linear Collider where broad resonances should occur in $e^+e^- \to \mu^+\mu^-$, see figure.

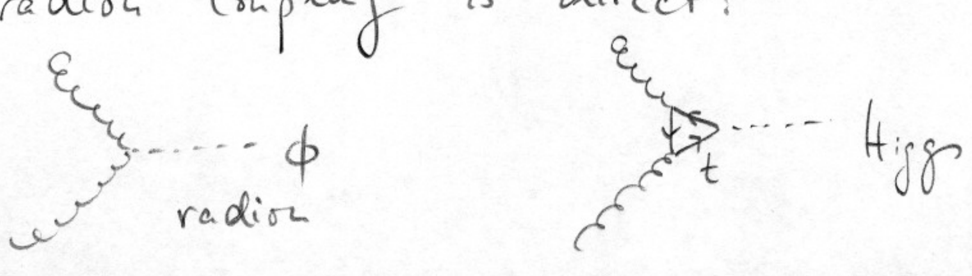
After radius stabilization the radion acquires a mass

$$m_\phi \sim \underbrace{\varepsilon}_{\mathcal{O}(1)} \frac{1}{\sqrt{kL}} M_{Pl}\, e^{-kL} .$$

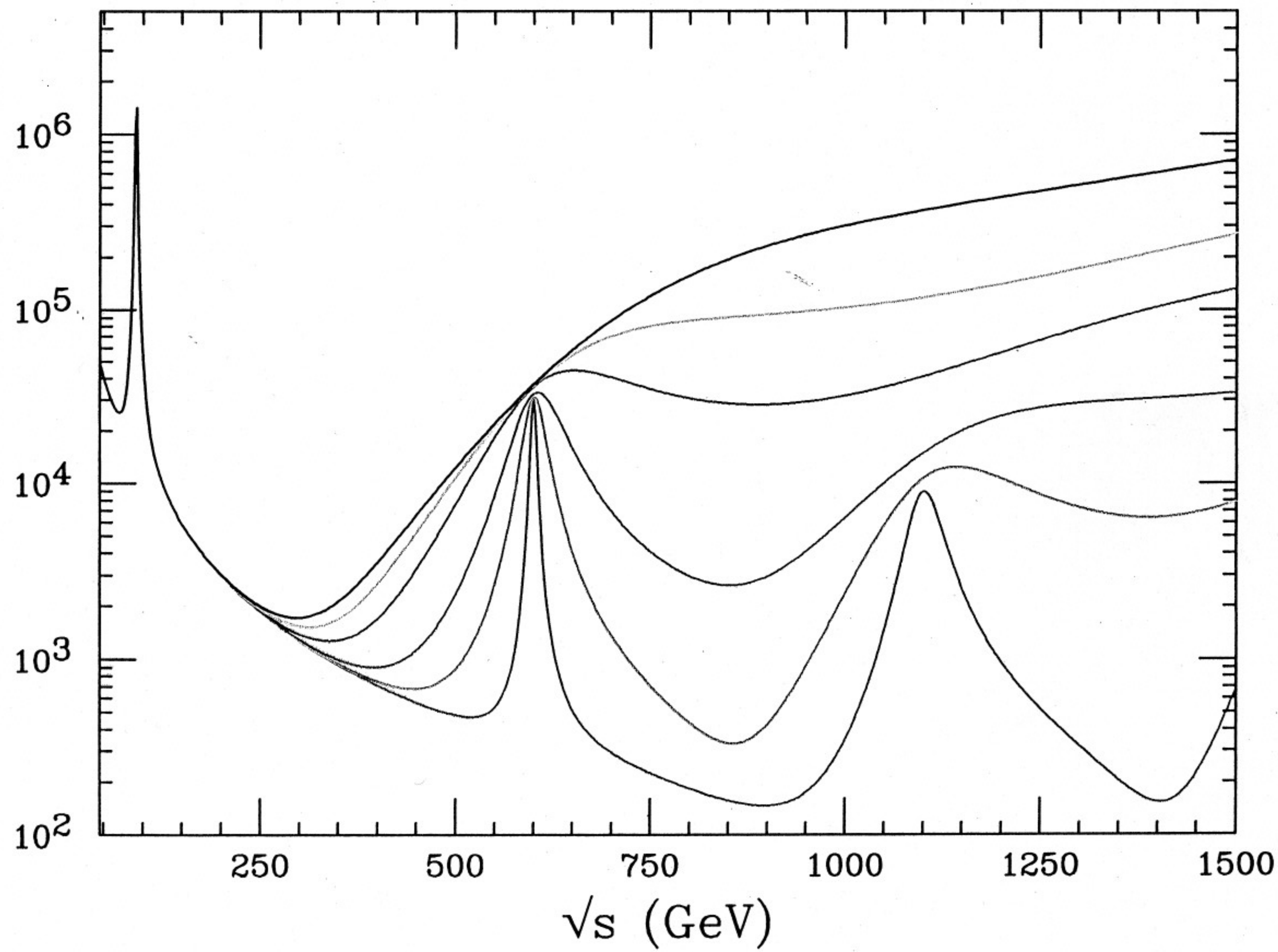
and the radion couplings are found to be

$$\frac{\phi}{\sqrt{6}\, M_{Pl}\, e^{-kL}}\, T^{\mu}{}_{\mu}$$

Many effects of the radion are very similar to those of a Higgs boson since they have similar couplings to matter. Radion and Higgs are therefore difficult to distinguish in RS1. Gluons, however, couple much stronger (~50 times) to radion than to Higgs, since the radion coupling is direct:



$\sigma(e^+e^- \to \mu^+\mu^-)$



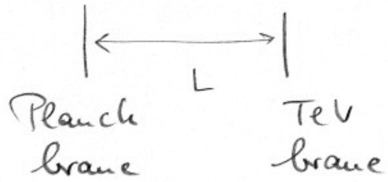
$m_1 = 600$ GeV assumed

and

$k/M_{Pl} = \begin{cases} 1.0 \\ 0.7 \\ 0.5 \\ 0.3 \\ 0.2 \\ 0.1 \end{cases}$

6) RS 2 model

In the RS 1 model the physical space was an interval $[0, L]$,

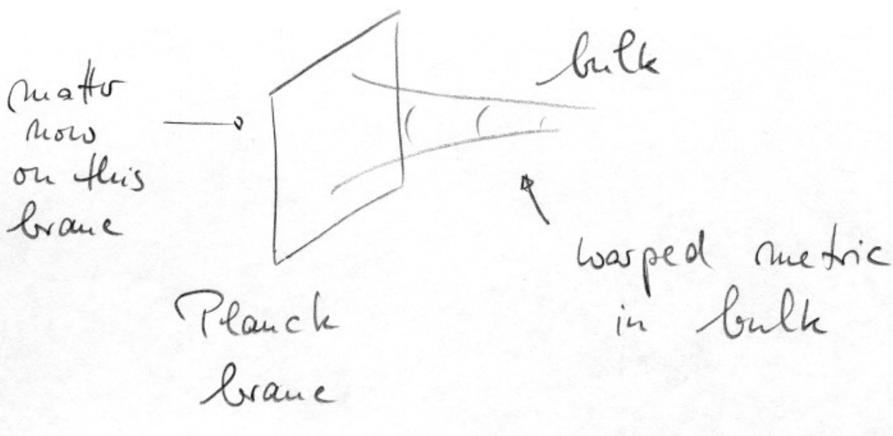


We found for the (effective) 4d Planck scale

$$M_{Pl}^2 = \frac{M_*^3}{k}\left(1 - e^{-2kL}\right).$$

Note that this depends only very little on L, and is even finite for $L \to \infty$! One could therefore consider the limit $L \to \infty$, taking the TeV brane to infinity – thus removing it completely.

Then there is only one (positive tension) brane left (the Planck brane), and we have an infinite extra dimension:



Naturally, matter now has to be put on the Planck brane.
This is the RS2 model.

Obviously, in contrast to the RS1 model, it is no longer designed to resolve the hierarchy problem. But it is still an interesting and simple setup.

In fact even the volume of the extra dimension remains finite.

$$V_5 = 2 \int d^4x \int_0^\infty dy \sqrt{|G|}$$
$$= V_4 \cdot 2 \cdot \int_0^\infty dy \; e^{-4ky}$$
$$= V_4 \frac{\ell}{2}$$

Hence $\ell = \frac{1}{k}$, and <u>not</u> L, plays the role of the compactification radius R_{comp} ($\rightarrow$ compare to flat extra dimensions).

Close to the brane, everything remains as it was in the RS 1 model.

In particular, the graviton zero-mode remains localized to the Planck brane, since it remains normalizable:
We had

$$\psi^{(0)} = e^{-\frac{3}{4}A(z)}$$

$$\rightarrow \int_0^{z_0} dz\, |\psi^{(0)}|^2 = \int_0^{z_0} dz\; e^{-3/2\, A(z)}$$

$$= \int_0^{z_0} dz\, \frac{1}{(1+k|z|)^{3/2}}$$

which is convergent in the limit $z_0 \to \infty$ ($\hat{=}\ L \to \infty$).

[Recall that in flat extra dimensions the zero-mode would become non-normalizable for $R_{comp} \to \infty$ due to the infinite volume of the flat extra dim and would then decouple.]

Again, the massive gravitons will stay away from the brane.
Therefore, normal 4d gravity is recovered on the Planck brane.

More precisely, corrections to the Newton potential in the effective 4d theory on the brane are of the form

$$V(r) = G_N \frac{M_1 M_2}{r} \left(1 + \frac{C}{(kr)^2}\right)$$

with $C \sim O(1)$. Since $k \sim M_{Pl}$ these corrections will be tiny.

One can even show that not only the Newton potential but also full 4d Einstein gravity is recovered on the brane in RS2.

Experimentally, it will be very difficult to distinguish RS2 from the SM. On the brane everything looks like the SM plus gravity. Therefore there is probably no chance to find limits for RS2 at colliders.

But there might be realistic possibilities to find limits for RS2 in astrophysics and (more likely) in cosmology.

7) AdS / CFT

It has been conjectured by Maldacena (and subsequently been confirmed in many tests) that gravity in AdS_5 space is equivalent to a certain gauge theory on 4d Minkowski space.
More precisely, type II B superstring theory on AdS_5 is equivalent to maximally supersymmetric Yang-Mills theory in 4d, according to that conjecture.
This supersymmetric theory in 4d is scale invariant and hence a conformal field theory (CFT).
The Maldacena conjecture has been extended in various ways and indeed seems to indicate much deeper relations between theories in different dimensions, in particular it relates theories with strong coupling to theories with weak coupling.

As the Randall - Sundrum models live exactly in AdS_5 the conjecture can be used to study these models. What emerges is an interesting holographic picture of warped extra dimensions, in which the physics in the bulk is "holographically" represented by the physics on the brane.

This holographic point of view has by now become very common in studies of RS models.

Details of the AdS / CFT correspondence are beyond the scope of the present lectures, but are highly recommended for further private study.