Extra Dimensions

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Extra Dimensions

1. Motivation

* Why do we live in 3+1 dimensions?
  → consider other possibilities in order to find answer

  Is there more than we can see?

* Kaluza, Klein: (1919 ... 1926)
  general relativity and $U(1)$ gauge theory can be unified in 5 dimensions!
  (did not work in original form, but maybe other possibilities ?!)
  → unification of gravity and gauge theory might require extra dimensions

* String theory (candidate for quantum gravity!) is consistent only in higher-dimensional space.
  5 consistent superstring theories found:
  type I, IIA, IIB, heterotic $\left\{ \begin{array}{l} E_8 \times E_8, \\ SO(32) \end{array} \right.$
  all only consistent in $d = 10$ dimensions

In string theory usually assumed
that 6 dimensions are compactified
on very small size manifold,
but mechanism for this not understood

→ try bottom-up approach!
→ field theory with additional dimensions

* Standard Model (SM) is very unlikely
to be the full story!
SM might work up to Planck scale M_{Pl}
($M_{Pl} \simeq 10^{19}$ GeV), at higher energy
quantum gravity should set in.
More likely appears grand unification
at about $M_{GUT} \simeq 10^{16}$ GeV, as suggested
by running couplings of SM, possibly
with supersymmetry (SUSY) added.
In any case M_{Pl} is natural cut-off
for SM physics.

Problems with SM: too many unknown
parameters
in particular: hierarchy problem!

→ Hierarchy problem:

* In SM the electroweak scale M_{EW} is set by the Higgs potential

$$V(\phi) = -\mu^2\phi^2 + \lambda\phi^4$$

→ Higgs VEV : $\frac{1}{\sqrt{2}} v$

$$v = \sqrt{\frac{\mu^2}{\lambda}} = 246 \text{ GeV}$$

Higgs mass $m_H = \sqrt{2}\,\mu$

Quantum corrections to Higgs mass are quadratically divergent:

$$\Delta\mu^2 \sim \Lambda^2_{cutoff}$$

→ If SM is valid up to M_{Pl} an enormous fine-tuning is required ("natural" value for m_H would be $\mathcal{O}(M_{Pl})$)

* usual solution to hierarchy problem:

SUSY !

with SUSY the corrections

+

have opposite sign

→ cancellation of quadratic divergences

With SUSY also GUT appears more likely. Still, the large ratio $\frac{M_{GUT}}{M_{EW}}$ remains unexplained.

Possible solution in context of extra dimensions: maybe in higher-dim. theory m_{EW} is the fundamental scale, and our 4d M_{Pl} appears large only because extra dimensions are large !?

* New ideas: (1980's, but mostly since 1998)

Extra dimensions might be much larger than M_{Pl}!

- naively would expect at most

$$r_{extra} \leq TeV^{-1} \simeq 10^{-18}\,m$$

from direct observation (absence of KK towers, see below)

- But: need not be true
  * large extra dimensions $R \simeq 0.01\,mm$ of Arkani-Hamed, Dimopoulos, Dvali (ADD)
  * warped extra dimensions of Randall and Sundrum (RS) even infinitely large possible

→ new activity in this field,
$\mathcal{O}(10^3) - \mathcal{O}(10^4)$ publications,
due to new possibilities in model building
(model building necessary in order to know what to look for at LHC, ...)
and in attempts to solve problems of the SM (hierarchy problem, SUSY breaking etc)

* important ingredient in most of these theories (models): branes.
Branes are lower-dimensional objects (p-brane has p space-like dimensions) in a higher-dim. space to which particles can be confined, or on which particles can be localized.
In string theory branes (in particular D-branes, Dirichlet branes) appear naturally as solutions of low energy (supergravity) equations. Open strings can end on such branes, hence particle modes of these strings are naturally confined to the brane.

* Variety of possibilities with large and small extra dimensions have been discovered:

  - large extra dimensions (ADD)
  - warped extra dimensions RS I, RS II
  - TeV scale extra dimensions, size $\sim 10^{-18}$ m
    (→ universal extra dimensions, UED)
  - GUT scale extra dimensions
  - deconstructed extra dimensions
    (dimensions emerging from dynamics of gauge theories)

  Not all address same problems, mechanisms of different models can often be combined

* <u>general picture</u>:

  effective theories in mind
  → renormalizability is not a problem
  (5d gauge theories in general not renormalizable)

* These lectures:

  - Introduction to main ideas of extra dimensions, main models, some interesting mechanisms

  - assumed knowledge:
    basics of general relativity
    QFT
    gauge theory, Standard Model
    some cosmology

  - no attempt at historical completeness, only few references

  For further reading (lecture notes, some original papers etc) see
  www.thphys.uni-heidelberg.de/~ewerz

2. Kaluza - Klein (KK) theory

Consider 3+1 - dim spacetime with coordinates

$$x = (x^0, \dots, x^3) = x^\mu$$

and metric $g_{\mu\nu}$, $\mu, \nu = 0, \dots, 3$.

We use sign convention $(+,-,-,-)$ for the metric, that is for flat metric we have

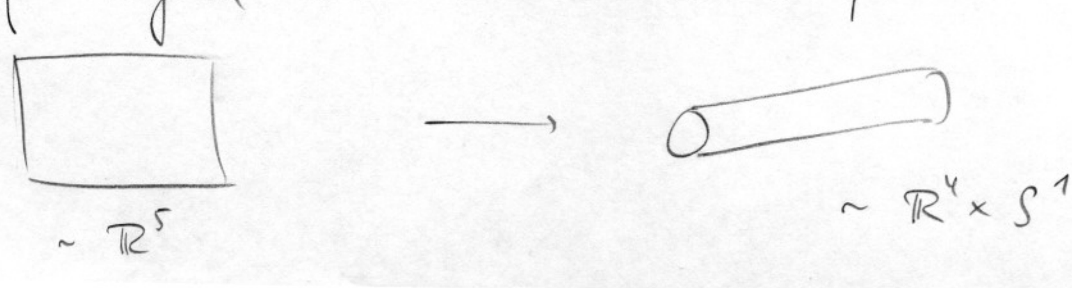
$$\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-).$$

Consider further a D-dimensional space-time of the form

$$M^4 \times X^{D-4},$$

that is a direct product of Minkowski spacetime M^4 with a compact ("internal") manifold X^{D-4} representing some extra dimensions. (We assume $M^4 \times X^{D-4}$ to be a solution of D-dim. Einstein equations.)

A simple example is a 5-dim. spacetime compactified on a circle (= periodic identification)



In the D-dim. spacetime we have coordinates

$$X = (x^0, \ldots, x^3, x^5, \ldots x^D) = X^A$$

and metric

$$g_{MN} \; (\text{or } G_{MN}) \quad M, N = 0, 1, \ldots, 3, 5, \ldots, D$$

and we use again sign conventions $(+, -, \ldots, -)$.

Let us now consider $M^4 \times S^1$, with

$$x^5 = y \in [0, 2\pi L[$$

with periodic boundary conditions.

* We start with a scalar field ϕ of mass m, 5-d. Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_A \phi \, \partial^A \phi - \frac{1}{2} m^2 \phi^2$$

From periodicity in x^5-direction:

$$\phi(x, y) = \phi(x, y + 2\pi L)$$

Expanding in harmonics on the circle:

$$\phi(x, y) = \sum_{n=-\infty}^{\infty} \phi_n(x) \, e^{iny/L}$$

to diagonalize ∂_5. Reality of ϕ requires

$$\phi_n^* = \phi_{-n} .$$

Substituting this in $\mathcal{L}$ we obtain for the action

$$S = \int d^4x \int_0^{2\pi L} dy \; \mathcal{L}$$

with the rescaling $\varphi_n = \sqrt{2\pi L}\, \phi_n$

$$S = \int d^4x \Big[\tfrac{1}{2} \partial_\mu \varphi_0 \partial^\mu \varphi_0 - \tfrac{1}{2} m^2 \varphi_0^2 + \sum_{n=1}^{\infty} \Big(\partial_\mu \varphi_n \partial^\mu \varphi_n^* - \Big(m^2 + \frac{n^2}{L^2}\Big) \varphi_n \varphi_n^* \Big) \Big]$$

Here we can read off the resulting 4d effective theory.

The spectrum is:

- a single real scalar field φ_0, a so-called zero-mode (which does not depend on y in the original 5d theory) – often the term "zero-mode" is used only for massless modes.
  We have mass m for φ_0.
- an infinite tower of massive complex scalars φ_n with masses

$$m_{eff}^2 = m^2 + \frac{n^2}{L^2}$$

These φ_n are called KK modes.

For small m ($m \to 0$) this implies an equidistant mass spectrum, $m_{eff} = \frac{n}{L}$.

More generally, compactification on a torus $T^\delta = \underbrace{S^1 \times \ldots \times S^1}_{\delta \text{ extra dimensions}}$

we find KK modes

$$\varphi_{\vec{n}} \quad \text{with} \quad \vec{n} = (n^5, \ldots, n^{4+\delta}) \in \mathbb{Z}^\delta$$

with masses

$$m_{eff}^2 = m^2 + \left(\frac{\vec{n}}{L}\right)^2 .$$

Let us now turn to the original model of Kaluza and Klein, which shows that 4d Einstein gravity and electrodynamics can be unified in a 5d theory.

Consider 5d Einstein-Hilbert action

$$S = \frac{1}{2} M_*^3 \int d^4x \, dy \, \sqrt{G} \, R^{(5)}$$

on the space $M^4 \times S^1$.

Here M_* is the fundamental Planck scale in 5d (M_*^3 occurs to make S dimensionless, $\hbar=1$).

$R^{(5)}$ is the 5d scalar curvature.

[Aside: short (!) summary of general relativity:

metric $g_{\mu\nu}$ on manifold M in 4d

→ obtain Levi-Civita connection with Christoffel symbols $\Gamma^{\sigma}_{\lambda\mu}$ related to $g_{\mu\nu}$ via

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left[\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right]$$

then curvature tensor $R^{\lambda}{}_{\mu\nu\kappa}$ is

$$R^{\lambda}{}_{\mu\nu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu} \Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\lambda}_{\nu\eta} .$$

Get Ricci tensor $R_{\mu\kappa}$

$$R_{\mu\kappa} = R^{\lambda}{}_{\mu\lambda\kappa}$$

and scalar curvature R

$$R = g^{\mu\kappa} R_{\mu\kappa}$$

Then Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G_N T_{\mu\nu}$$

$G_N = \frac{1}{8\pi M_{Pl}^2}$

energy momentum tensor $T_{\mu\nu}$

Equivalent Einstein-Hilbert action is

$$S = \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{g}\, R$$

↑ det $g_{\mu\nu}$

L

Expand G_{AB} in harmonics on the circle:

$$G_{AB}(x,y) = \sum_{n=-\infty}^{\infty} G_{AB}^{(n)} \, e^{iny/L}$$

For now concentrate only on zero-mode $G_{AB}^{(0)}$ and neglect all massive modes.

We decompose G_{AB} as

$$G_{AB}^{(0)} = \left(\begin{array}{c|c} G_{\mu\nu}^{(0)} & G_{\mu 5}^{(0)} \\ \hline G_{5\mu}^{(0)} & G_{55}^{(0)} \end{array} \right)$$

and write

$$G_{\mu\nu}^{(0)} = e^{\phi/\sqrt{3}} \left[g_{\mu\nu}(x) + e^{-\sqrt{3}\phi} A_\mu A_\nu \right]$$

$$G_{\mu 5}^{(0)} = G_{5\mu}^{(0)} = e^{-2\phi/\sqrt{3}} A_\mu$$

$$G_{55}^{(0)} = e^{-2\phi/\sqrt{3}}$$

That gives for the 4d effective action (with only the zero-mode fields)

$$S_{\text{zero-mode}} = M_*^3 \pi L \int d^4x \sqrt{g} \left[R^{(4)} + \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{4} e^{-\sqrt{3}\phi} F_{\mu\nu}^2 \right]$$

Note that in the above ds_5^2 is invariant under the gauge transformation

$$g_{\mu\nu} \to g_{\mu\nu} \quad , \quad \phi \to \phi \quad , \quad A_\mu \to A_\mu + \partial_\mu \alpha$$

Assuming constant ϕ we thus have 4d gravity and a $U(1)$ gauge field. (ϕ actually corresponds to size of extra dim., but ϕ does not have a potential $\rightarrow$ problems with this theory.)
Comparing with conventional 4d Einstein-Hilbert action we find that

$$M_{Pl}^2 = M_*^3 \, 2\pi L$$

or for the Newton constant

$$G_N = \frac{1}{8\pi M_{Pl}^2} = \frac{1}{16\pi^2 L M_*^3} \, .$$

The massive KK levels give a massive graviton with mass $m_n^2 = \frac{n^2}{L^2}$ at each (n'th) level. Their mass comes due to Higgs mechanism: one massless graviton (2 d.o.f.) eats one massless gauge field (2 d.o.f.) and one real scalar (1 d.o.f.) to make a massive 4d graviton (5 d.o.f.).
Massive KK modes are charged under the massless gauge field (charges $\sim q_n \sim \frac{m_n}{M_{Pl}}$).
At linearized level gauge transfs do not mix different KK levels, but do at nonlinear gravity level.

With more complicated internal spaces X^{D-4} one can also obtain non-Abelian gauge theories. However, then X^{D-4} is in general curved, hence Λ-term is required in order that $M^4 \times X^{D-4}$ is solution of Einstein equations. → also M^4 curved, hence more complications - not to be discussed here!

Up to energies of ~ 1 TeV no KK towers have been observed.

→ such a scenario does not allow large extra dimensions.

→ We next turn to models with branes.

3. Large extra dimensions (ADD model)

1) Simple picture

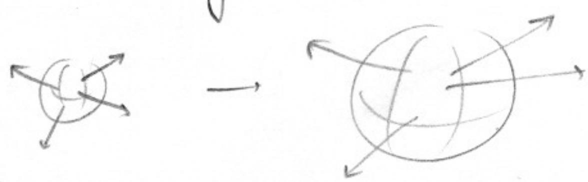
Recall hierarchy problem: $M_{EW} \ll M_{PL}$

→ Why is gravity so weak?

- electric force between two electrons $\sim 10^{43}$ times stronger than gravitational force!
  These forces would be equal if electron had 10^{22} times more mass, that is had $\sim M_{PL} = 10^{19}$ GeV $\hat{=}$ 10^{-35} m.

In 3 space dimensions grav. force is

$F \sim \frac{1}{R^2}$ (since surface $\sim R^2$



)

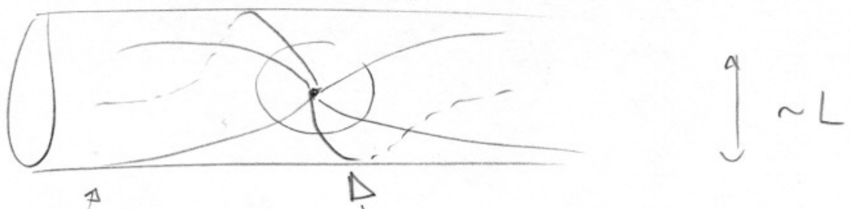
while in 4 space dim.

$F \sim \frac{1}{R^3}$ (surface $\sim R^3$)

→ If gravity has more dimensions to propagate in, force becomes strong already at larger distance.

→ unification at M_{EW} instead of M_{PL}?!

In a model with one compactified extra dimension this gives

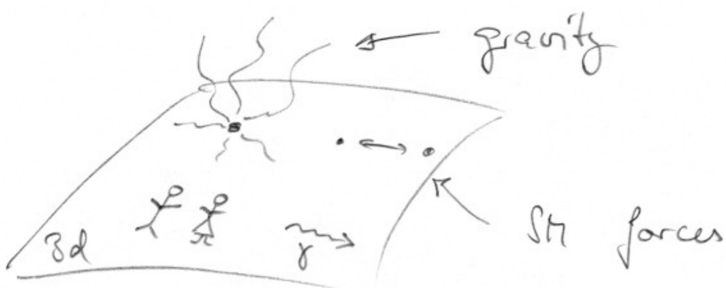


here $F \sim \frac{1}{R^2 L}$ here $F \sim \frac{1}{R^{2+\delta}}$

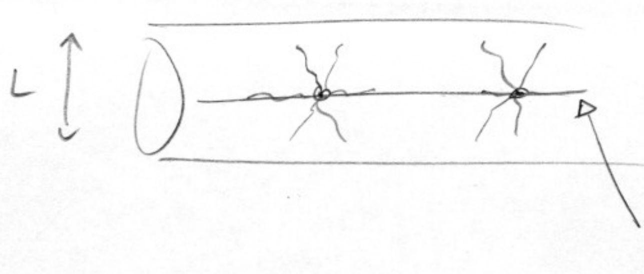
$\delta = 1$ for one extra dim.

Why does this not apply to electroweak & strong interactions?

→ SM particles and interactions confined to a brane!



or with compactified dim.



2) Matching higher dimensional to 4d theory

How do we obtain the mass scales and couplings in the 4d effective theory from the higher-dimensional fundamental theory?

Consider higher-dim theory with Planck scale M_* and compact extra dimensions of size r. Metric tensor g_{AB} has mass dimension $[g] = 0$, Christoffel symbol $[\Gamma] = 1$, Ricci tensor and scalar curvature $[R_{MN}] = 2$, $[R] = 2$, all independent of total number of dimensions.

Einstein-Hilbert action in $4+n$ dimensions:

$$S_{4+n} = -M_*^{n+2} \int d^{4+n}x \sqrt{g^{(4+n)}}\, R^{(4+n)}$$

↑ such that S_{4+n} dimensionless ($\hbar = 1$)

As in example of KK model we have to match this to 4d action

$$S_4 = -M_{Pl}^2 \int d^4x \sqrt{g^{(4)}}\, R^{(4)}$$

For this knowledge of extra-dim. geometry required.

We now assume flat spacetime and n compact extra dimensions, hence

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu - r^2\, d\Omega^2_{(n)}$$

with $h_{\mu\nu}$ 4d fluctuation around flat 4d Minkowski metric $\eta_{\mu\nu}$. This tells us how 4d graviton ($h_{\mu\nu}$) is contained in higher-dim. action.

From this we get

$$\sqrt{g^{(4+n)}} = r^n \sqrt{g^{(4)}}$$
$$R^{(4+n)} = R^{(4)}$$
calculated from $h_{\mu\nu}$.

Hence

$$S_{4+n} = - M_*^{n+2} \underbrace{\int d\Omega_{(n)}\, r^n}_{= V_{(n)}} \int d^4x \sqrt{g^{(4)}}\, R^{(4)} \,,$$

$= V_{(n)}$ volume of extra-dim. space

for torus compactification

$$V_{(n)} = (2\pi r)^n$$

Hence

$$M_{Pl}^2 = V_{(n)} M_*^{n+2} = (2\pi r)^n M_*^{n+2}$$

→ a large volume of the extra dim. can induce a large M_{Pl} although M_* is not large!

Similarly, the matching for gauge fields can be done. In $(4+n)$ d (gauge fields in all dimensions!)

$$S^{(4+n)} = -\int d^{4+n}x \, \frac{1}{4g_*^2} F_{MN} F^{MN} \sqrt{g^{(4+n)}}$$

with higher dim. field strength tensor F_{MN} and $(4+n)$ d gauge coupling g_*.
Concentrating on how 4d field strength $F_{\mu\nu}$ are contained in F_{MN}, we can integrate over extra dim.:

$$S^{(4)} = -\int d^4x \, \frac{V_{(n)}}{4g_*^2} F_{\mu\nu} F^{\mu\nu} \sqrt{g^{(4)}} + \ldots$$

Thus couplings are related by

$$\frac{1}{g_{eff}^2} = \frac{V_{(n)}}{g_*^2} .$$

Note that higher-dim. coupling g_* hence has to have mass dimension (since $[g_{eff}] = 0$)

$$[g_*] = -\frac{n}{2} .$$

Consequently, higher-dim. gauge theory cannot be renormalizable.
$\rightarrow$ has to be considered as effective theory of some more fundamental theory at even higher energy (strings?)

Natural assumption is that scale of $(4+n)d$ gauge coupling is also M_*,

$$g_* \sim \frac{1}{M_*^{n/2}}$$

Hence

$$\frac{1}{g_4^2} = V_{(n)} M_*^n \sim r^n M_*^n$$

$$M_{Pl}^2 = V_{(n)} M_*^{n+2} \sim r^n M_*^{n+2}$$

$$\rightarrow \quad r \sim \frac{1}{M_{Pl}} g_4^{-\frac{n+2}{n}}$$

For $g \sim \mathcal{O}(1)$ the "natural" size of the extra dimensions is hence $r \sim \frac{1}{M_{Pl}}$.
This was the general opinion until the 1990's. Note that above relation results from assuming gauge fields to propagate in all dimensions. If they were confined to only 4d then only the strength of gravity (M_{Pl} vs. M_*) would be affected.

3) How large can "large" possibly be?

At distances below the compactification radius we expect deviations from $\frac{1}{r}$ - potential for gravity. Actual experiments can now measure gravity at distances down to ~ 0.1 mm, and so far no deviation has been found. Hence we have the bound

$$r \leq 0.1 \text{ mm}$$

Since $M_*^{n+2} \sim \frac{M_{Pl}^2}{r^n}$, $(4+n)$d Planck scale is lowered by the extra dimensions if $r > \frac{1}{M_{Pl}}$.

$M_* \lesssim 1$ TeV is excluded from the non-observation of strong gravity effects at colliders so far.

→ lowest possible value is ~1 TeV, thus

$$M_* \geq 1 \text{ TeV}.$$

Such models with $M_* \sim O(1 \text{ TeV})$ are called "large extra dimensions".

(Arkani-Hamed, Dimopoulos, Dvali, '98,
also Antoniadis '90)

Tests of the Gravitational Inverse-Square Law below the Dark-Energy Length Scale

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(Dated: November 30, 2007)

We conducted three torsion-balance experiments to test the gravitational inverse-square law at separations between 9.53 mm and 55 μm, probing distances less than the dark-energy length scale $\lambda_d = \sqrt[4]{\hbar c/\rho_d} \approx 85\ \mu$m. We find with 95% confidence that the inverse-square law holds ($|\alpha| \leq 1$) down to a length scale $\lambda = 56\ \mu$m and that an extra dimension must have a size $R \leq 44\ \mu$m.

PACS numbers: 04.80.-y,95.36.+x,04.80.Cc,12.38.Qk

Recent cosmological observations[1, 2, 3] have shown that 70% of all the mass and energy of the Universe is a mysterious "dark energy" with a density $\rho_d \approx 3.8$ keV/cm^3 and a repulsive gravitational effect. This dark-energy density corresponds to a distance $\lambda_d = \sqrt[4]{\hbar c/\rho_d} \approx 85\ \mu$m that may represent a fundamental length scale of gravity[4, 5]. Although quantum-mechanical vacuum energy should have a repulsive gravitational effect, the observed ρ_d is between 10^{60} to 10^{120} times smaller than the vacuum energy density computed according to the standard laws of quantum mechanics. Sundrum[6] has suggested that this huge discrepancy (the "cosmological constant problem") could be resolved if the graviton were a "fat" object with a size comparable to λ_d that would prevent it from "seeing" the short-distance physics that dominates the vacuum energy. His scenario implies that the gravitational force would *weaken* for objects separated by distances $s \lesssim \lambda_d$. Dvali, Gabadaze and Senjanovíc[7] argue that a similar weakening of gravity could occur if there are extra *time* dimensions. In their scenario, the standard model particles are localized in "our" time, while the gravitons propagate in the extra time dimension(s) as well. Other scenarios predict the opposite behavior: the extra *space* dimensions of M-theory would cause the gravitational force to get *stronger* for $s \lesssim R$ where R is the size of the largest compactified dimension[8]. These considerations, plus others involving new forces from the exchange of proposed scalar or vector particles[9] motivated the tests of the gravitational inverse-square law we report in this Letter.

Our tests were made with a substantially upgraded version of the "missing mass" torsion-balance instrument used in our previous inverse-square-law tests[10, 11]. The instrument used in this work[12], shown in Fig. 1, consisted of a torsion-pendulum detector suspended by a thin $\approx$ 80-cm-long tungsten fiber above an attractor that was rotated with a uniform angular velocity ω by a geared-down stepper motor. The detector's 42 test bodies were 4.767-mm-diameter cylindrical holes machined into a 0.997-mm-thick molybdenum detector ring. The hole centers were arrayed in two circles, each of which had 21-fold azimuthal symmetry. The attractor had a similar 21-fold azimuthal symmetry and consisted of a 0.997 mm thick molybdenum disc with 42 3.178-mm-diameter holes mounted atop a thicker tantalum disc containing 21 6.352-mm-diameter holes. The gravitational interaction between the missing masses of the detector and attractor holes applied a torque on the detector that oscillated 21 times for each revolution of the attractor, giving torques at 21ω, 42ω, 63ω, etc. that we measured by monitoring the pendulum twist with an autocollimator system. The holes in the lower attractor ring were displaced azimuthally by 360/42 degrees and were designed to nearly cancel the 21ω torque *if the inverse-square law holds.* On the other hand, an interaction that violated the inverse-square law, which we parameterize as a single Yukawa

$$V(r) = -G\frac{m_1 m_2}{r}[1 + \alpha \exp(-r/\lambda)] , \quad (1)$$

would not be appreciably canceled if λ is less than the 1 mm thickness of the upper attractor disc. We minimized electromagnetic torques by coating the entire detector with gold and surrounding it by a gold-coated shield consisting of a tightly-stretched, 10 μm-thick, beryllium-copper membrane between the detector and attractor plus a copper housing that had small holes

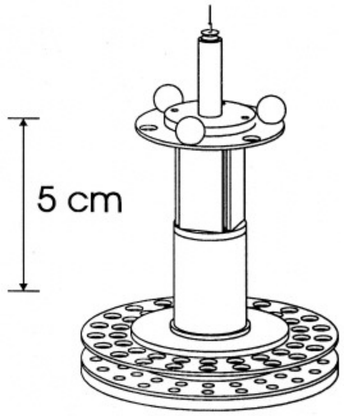
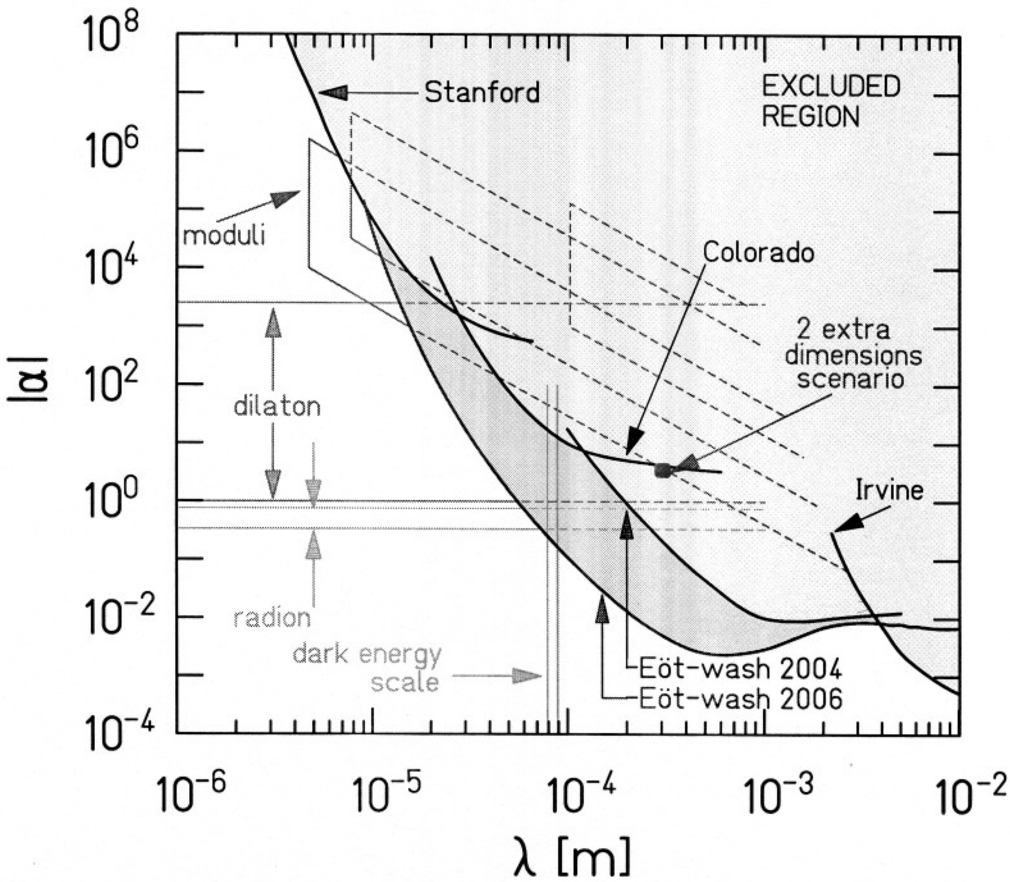


FIG. 1: Scale drawing of our detector and attractor. The 3 small spheres near the top of the detector were used for a continuous gravitational calibration of the torque scale. Four rectangular plane mirrors below the spheres are part of the twist-monitoring system. The detector's electrical shield is not shown.

from:

Kapner et al, 2006



for assumed gravity potential

$$V(r) = -G \frac{m_1 m_2}{r} \left[1 + \alpha \exp\left(-r/\lambda\right) \right]$$

Assuming $M_* \sim 1\,\text{TeV}$, which radius is required for n extra dimensions?

We have

$$\frac{1}{r} = M_* \left(\frac{M_*}{M_{Pl}}\right)^{\frac{2}{n}} = (1\,\text{TeV}) \cdot 10^{-\frac{32}{n}}$$

for $M_* = 10^3\,\text{GeV}$, $M_{Pl} = 10^{19}\,\text{GeV}$

or

$$r \simeq 2 \cdot 10^{-19}\; 10^{\frac{32}{n}}\; m$$

For $\underline{n=1}$: $r = 10^{13}\,m$ clearly excluded!

$\underline{n=2}$: $r \simeq 2\,mm$ now excluded

but: $M_* \gtrsim 3.2\,\text{TeV}$ still possible!

$\underline{n=3}$: $r \sim 10^{-8}\,m$ allowed

Note that in the ADD scenario the hierarchy problem is not really resolved but only translated into the problem of explaining why the extra dimensions are so large.

4) Brane and brane action

Branes have natural origin in string theory or as domain walls. In string theory, D-branes emerge as soliton solutions of equations of motion in supergravity approximation.

Here we will use low energy description: branes are assumed to be where they are without asking for their dynamical origin. At higher energies ($> \Lambda_{cut\text{-}off}$), however, dynamics of the brane will be important.

Consider 3-brane $\sim \mathbb{R}^4$ and additional dimensions X^n $(= \mathbb{R}^n, T^n, \dots)$

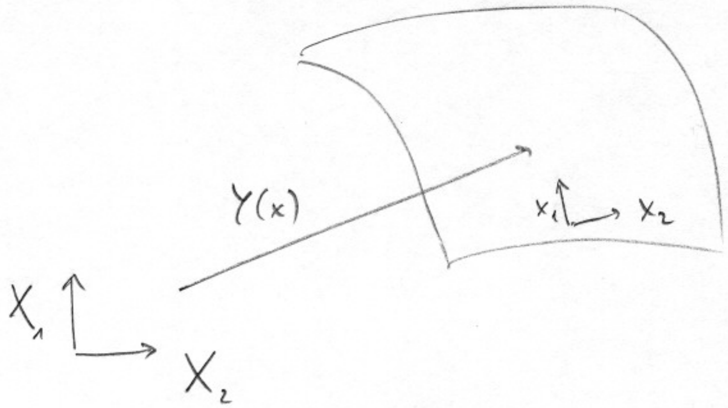
Coordinates in full space X^M, $M = 0, 1, \dots, 4+n$

" on the brane x_μ, $\mu = 0, 1, 2, 3$

" along extra dim. x_m, $m = 5, \dots, 4+n$

metric in higher dim. space $G_{MN}(X)$

Let position of the brane in higher-dim. space be $Y^M(x)$, and we have fields $\phi(x)$, $A_\mu(x)$, $\psi_L(x)$, ... on the brane.



Assuming flat brane in flat space as the vacuum,

$$G_{MN}(X) = \eta_{MN}$$
$$Y^M(x) = \delta^M_\mu x^\mu .$$

We have bulk ($\equiv$ total space) action

$$S_{bulk} = -\int d^{4+n}X \sqrt{|G|} \left(M_*^{n+2} R^{(4+n)} + \Lambda \right)$$

possible bulk cosmological constant

For effective action on the brane we need induced metric:

$$d^2s = G_{MN}\, dY^M(x)\, dY^N(x)$$
$$= G_{MN} \frac{\partial Y^M}{\partial x^\mu} dx^\mu \frac{\partial Y^N}{\partial x^\nu} dx^\nu$$

In flat case obtain $\eta_{\mu\nu}$ (flat 4d Mink. metric)

Now consider brane induced part of action. Needs to be invariant under general coordinate transformations of bulk coord. X and under general coord. transf.s of brane coord. x. The latter corresponds to reparametrisations of brane (surface) which cannot have physical significance.
→ will get 4d Lorentz invariance on brane.
(Technically, this requires to contract all bulk indices among themselves, and similarly for brane coord. indices.)
Hence general form of brane action:

$$S_{brane} = \int d^4x \sqrt{|g|} \Big[-f^4 - R^{(4)} + \frac{g^{\mu\nu}}{2} D_\mu \phi D_\nu \phi - V(\phi) - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \dots \Big]$$

f^4 is possible energy density on brane = brane tension

We assume brane tension small to be able to neglect back-reaction on background.
(Will be different later in RS scenario.)

4d reparametrisation invariance $x \to x'(x)$ requires four additional gauge fixing conditions. Possible choice

$$Y^\mu(x) = x^\mu$$

$\to$ only $Y^m(x)$, $m = 5, \dots, 4+n$, correspond to physical degrees of freedom.

Consider fluctuations of bulk metric G_{MN} around flat space background:

$$G_{MN} = \eta_{MN} + \frac{1}{2M_*^{n/2+1}} h_{MN} ,$$

such that graviton h_{MN} has dimension $\frac{n}{2}+1$, as is usual for boson in $4+n$ dimensions. Factor $\frac{1}{2}$ required to get canonically normalized kinetic term when expanding Einstein-Hilbert action in h.

Expanding leading term in brane action ($\leftarrow$ coord. Y^M of brane),

$$\int d^4x \sqrt{|g|} \left[-f^4 + \dots \right]$$

with $g_{\mu\nu} = G_{MN} \partial_\mu Y^M \partial_\nu Y^N = \eta_{\mu\nu} + \partial_\mu Y^m \partial_\nu Y_m$

and $\det g = -\partial_\mu Y^m \partial^\mu Y_m$

gives $S = \int d^4x \, f^4 \, \partial_\mu Y^m \partial^\mu Y_m$

Note that negative tension $\tau = f^4$ would give negative kinetic energy term for Y^m, hence a ghost – signalling instability of a negative tension brane configuration.
(However, projecting out negative kinetic energy parts is possible → orbifolds.)

5) Coupling of SM fields to bulk gravitons

How does matter (SM) on the brane interact with the various graviton modes?
→ construct interaction Lagrangian
SM fields see only induced metric

$$g_{\mu\nu}(x) = G_{MN}(x)\, \partial_\mu Y^M \partial_\nu Y^N$$

Let us now concentrate on bulk graviton modes and set brane fluctuations to zero, that is $Y^M = \delta^M_\mu x^\mu$, $Y^m = 0$.
Then

$$g_{\mu\nu}(x) = G_{\mu\nu}(x_\mu, x^m = 0)$$

Brane action is

$$S = \int d^4x \sqrt{|g|}\; \mathcal{L}_{SM}(g_{\mu\nu}, \phi, A, \psi, \dots)$$

Writing arguments explicitly,

$$S_{SM}[g, \phi, \dots] = \int d^4x \sqrt{|g|}\; \mathcal{L}_{SM}(g, \phi, \dots)$$

Expanding around flat induced metric:

$$S_{SM}[g, \phi, \dots] = \int d^4x\; \mathcal{L}_{SM}(\eta, \phi, \dots)$$

$$+ \int d^4x \left.\frac{\delta S_{SM}}{\delta g_{\mu\nu}(x)}\right|_{g=\eta} \delta g_{\mu\nu}(x) + \dots$$

with

$$\delta g_{\mu\nu}(x) = \frac{1}{M_*^{\frac{n}{2}+1}} h_{\mu\nu}(x)$$

Now

$$T^{\mu\nu}_{SM} = \frac{1}{\sqrt{|g|}} \left.\frac{\delta S_{SM}}{\delta g_{\mu\nu}(x)}\right|_{g=\eta}$$

is energy-momentum tensor of SM matter.

→ interaction of SM matter to graviton is

$$S_{int} = \int d^4x\; T^{\mu\nu} \frac{h_{\mu\nu}(x)}{M_*^{\frac{n}{2}+1}}$$

Note linear coupling of graviton to energy-momentum tensor. (Actually, this can be viewed as definition of $T^{\mu\nu}$)

In above $h_{\mu\nu}(x)$ is superposition of KK modes. For toroidal compactification of extra dim. with volume $V_n = (2\pi R)^n$ we can write

$$h_{MN}(x,y) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \frac{h^{\vec{k}}_{MN}(x)}{\sqrt{V_n}} \, e^{i\vec{k}\cdot\vec{y}/R}$$

$y^n = x^n$ for extra dim.

Hence coupling of SM matter to individual KK modes:

$$\sum_{\vec{k}} \int d^4x \, T^{\mu\nu} \frac{1}{M_*^{\frac{n}{2}+1}} \frac{h^{\vec{k}}_{\mu\nu}}{\sqrt{V_n}} = \sum_{\vec{k}} \int d^4x \, \frac{1}{M_{Pl}} T^{\mu\nu} h^{\vec{k}}_{\mu\nu}$$

Thus coupling of each individual mode is of strength $\frac{1}{M_{Pl}}$. But since there are many of those modes, the total coupling to the bulk graviton is of strength $\frac{1}{M_*}$! (see above action before KK decomposition)

A closer look at different degrees of freedom of graviton:

graviton is $D \times D$ symmetric tensor, $D = n+4$, hence $\frac{1}{2} D(D+1)$ components.