Hydrodynamic attractor in periodically driven ultracold quantum gases

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Hydrodynamic attractors are a universal phenomenon of strongly interacting systems that describe the hydrodynamic-like evolution far from local equilibrium. In particular, the rapid hydrodynamization of the Quark-Gluon Plasma is behind the remarkable success of hydrodynamic models of high-energy nuclear collisions. So far, hydrodynamic attractors have been explored only in systems undergoing monotonic expansion, such as Bjorken flow. We demonstrate that a system with an oscillating isotropic expansion exhibits a novel cyclic attractor behavior. This phenomenon can be investigated in ultracold quantum gases with externally modulated scattering length, offering a new avenue for experimentally discovering hydrodynamic attractors.

Introduction.—The study of how physical systems approach equilibrium is a recurring question in many branches of physics, from cosmology [1, 2] to ultracold atoms [3–5] to high-energy nuclear collisions [6–8]. In the context of QCD matter thermalization, an essential role is played by nonthermal [9] and hydrodynamic [10] attractors characterized by the rapid loss of initial state information. While viscous (Navier-Stokes) hydrodynamics is the universal long distance and long time description near equilibrium [11], the *hydrodynamic attractor* refers to the hydrodynamic-like behavior when perturbations from equilibrium are still significant. The remarkable success of hydrodynamics models in describing the expansion of Quark Gluon Plasma created in high-energy nuclear collisions is credited to such rapid hydrodynamization [6–8].

Hydrodynamic attractors have been studied in many theoretical contexts: holography [10, 12–16], QCD Kinetic Theory [17–21], Relativistic Boltzmann Transport [22, 23], Boltzmann RTA [24–30], and generalized hydrodynamic theories [25, 31–36]; see also review articles [37–40]. Hydrodynamic attractors have a rich mathematical structure based on transseries [29, 41, 42], and can be viewed more generally as dynamical dimensionality reduction of phase space [43]. Recently, a proposal to measure attractor behavior with ultracold atoms near unitarity was presented in Ref. [44]. However, so far attractor behavior has been studied only in monotonically expanding or driven systems. In this Letter, we present the first study of hydrodynamic attractors in *periodically* driven systems, which are especially relevant for ultracold atom experiments.

Hydrodynamic attractor behavior was first discovered in modeling the equilibration of high-temperature QCD matter in high-energy heavy-ion collisions. Right after the two nuclei pass each other, the system expands rapidly along the beam axis [45]. The expansion rate is $\partial_\mu u^\mu = 1/\tau$, where u^μ is a 4-velocity and τ is the proper time after the collision. High-temperature QCD matter is approximately conformal, and Navier-Stokes hydrodynamics predicts that the expansion induces an anisotropy in the energy-momentum tensor proportional to the shear viscosity η [38],

$$\frac{T^{xx}_{\rm NS} - P_{\rm eq}}{P_{\rm eq}} = \frac{2}{3}\frac{\eta}{\tau P_{\rm eq}}\,. \tag{1}$$

The deviation from the equilibrium pressure $P_{\rm eq}$ is controlled by the dimensionless time variable $\tilde{w} = \tau P_{\rm eq}/(\pi\eta) = \tau T/(4\pi\eta/s)$ [24], which is related to temperature T and the celebrated shear viscosity over entropy ratio η/s [46]. For $\tilde{w} \to \infty$ the viscous corrections vanish and the system becomes isotropic. However, at early times $\tilde{w} \to 0$, the viscous corrections $1/\tilde{w}$ in Eq. (1) are large and Navier-Stokes hydrodynamics becomes invalid.

The early-time equilibration dynamics have been studied in many microscopic theories [37–40]. These studies reveal that the full pressure anisotropy $(T^{xx} - P_{\rm eq})/P_{\rm eq}$ converges to the Navier-Stokes expectation already for $\tilde{w} = \mathcal{O}(1)$ when viscous corrections are still large. Furthermore, for different initial conditions the pressure anisotropy relaxes to a common trajectory even before reaching the Navier-Stokes expectation. This emergent far-from-equilibrium hydrodynamic behavior is known as the hydrodynamic attractor.

In this paper, we show that systems undergoing periodic expansion and contraction never relax to a Navier-Stokes limit. Instead, they settle for a new type of cyclic hydrodynamic attractor. For such systems, the expected deviation from equilibrium, Eq. (1), vanishes at certain turning points, hence the scaled time $\tilde{w}$ is not a meaningful time variable. Instead, we propose to plot the actual anisotropy $(T^{xx} - P_{\rm eq})/P_{\rm eq}$ versus the Navier-Stokes expectation $(T^{xx}_{\rm NS} - P_{\rm eq})/P_{\rm eq}$. Deviations from the diagonal $y = x$ line serve as a clear indicator of departures from Navier-Stokes hydrodynamics.

For many years, the equation of state and transport coefficients of ultracold quantum fluids have been studied via the collective mode excitations in a harmonic trapping potential. Kinetic theory computations provide results that interpolate between the collisionless and hydrodynamic limits [47–49]. In particular, the damping of the radial breathing mode is determined by the bulk viscosity ζ. The bulk viscosity vanishes in a nonrelativistic,

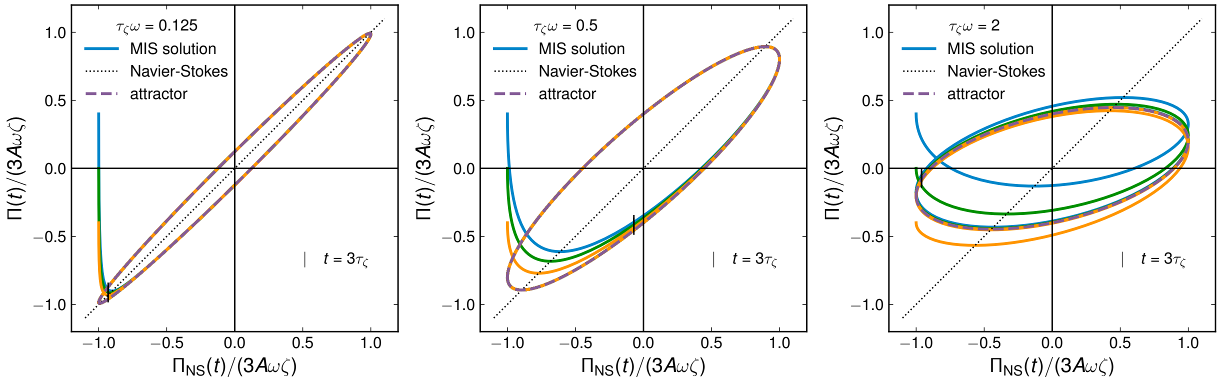

FIG. 1. Müller-Israel-Stewart hydrodynamics: attractor solution (limiting cycle) for oscillating expansion at different drive frequencies ω. The full dynamical bulk pressure $\Pi(t) \equiv \frac{1}{3}T^i_{\ i} - P_{\rm eq}$ is plotted vs its Navier-Stokes approximation $\Pi_{\rm NS}(t)$, such that deviations from Navier-Stokes appear outside the diagonal (dotted line). Different initial conditions (colors) converge toward an asymptotic attractor solution (dashed ellipse) near $t = 3\tau_\zeta$, as marked by the vertical bar. For increasing drive frequency (left to right panels) the elliptic shape of the out-of-equilibrium attractor widens; this signifies a growing deviation from Navier-Stokes.

scale invariant gas [50–54], and this has been confirmed experimentally for the three-dimensional resonantly interacting Fermi gas [55, 56]. In two dimensions, instead, scale invariance is broken by quantum fluctuations [57–61] and a nonzero damping is predicted, however it was too small to observe in earlier experiments [62, 63].

Recent works demonstrate a new and more sensitive way to probe expansion dynamics in dilute quantum gases: external modulation of the scattering length $a(t)$ is equivalent to fluid expansion and probes local dissipation even in a uniform system without fluid motion [44, 64]. Experimentally, continuous periodic modulation of the scattering length has been used to nonlinearly excite matter-wave jets [65] and for spontaneous pattern formation [66–68]; in two-dimensional Fermi gases it has been used to excite the amplitude ("Higgs") mode of the superfluid order parameter [69]. Attractor behavior, however, is predicted to appear already in linear response for a small-amplitude drive, and this regime can be probed with new detection schemes for the instantaneous equation of state.

In the following, we first present how hydrodynamic attractors arise in periodically driven systems. In linear response these can be described by Müller-Israel-Stewart hydrodynamics, which applies both to relativistic heavy-ion and nonrelativistic ultracold atom systems. Next, in order to study attractors in the nonlinear regime of large drive amplitude, we use relativistic kinetic theory for massive particles, which includes the ultracold atom case as the nonrelativistic limit.

Müller-Israel-Stewart theory.—We consider a three-dimensional Fermi gas in an isotropic trap whose inverse scattering length $a^{-1}(t)$ is modulated periodically over time. As shown in Ref. [64], the time variation of the scattering length leads to the same local dissipation as the homogeneous expansion of that gas with $\theta = \partial_\mu u^\mu = 3\partial_t \log a^{-1}(t)$. The linear response of the bulk pressure $\Pi(t) \equiv \frac{1}{3}T^i_{\ i} - P_{\rm eq}$ can be described by the Müller-Israel-Stewart (MIS) equation [70, 71]

$$\tau_\zeta \dot{\Pi}(t) = -\Pi(t) + \Pi_{\rm NS}(t), \qquad (2)$$

where τ_ζ is the relaxation time and $\Pi_{\rm NS}(t) = -\zeta[a(t)]\theta$ is the Navier-Stokes expectation.

The general solution of Eq. (2) for arbitrary drive $\Pi_{\rm NS}(t)$ starting at $t = 0$ is given by

$$\Pi(t) = e^{-t/\tau_\zeta}\Pi(0) + \frac{1}{\tau_\zeta}\int_0^t dt'\, e^{(t'-t)/\tau_\zeta}\Pi_{\rm NS}(t'). \qquad (3)$$

Consider a periodic drive in the linear regime

$$a^{-1}(t) = a_0^{-1}(1 + A\sin\omega t), \quad A \ll 1. \qquad (4)$$

Here we consider a finite a_0^{-1}. We note that in ultracold gases it is also possible to modulate the scattering length around the scattering resonance at $a_0^{-1} = 0$ because the bulk viscosity is suppressed near the resonance as $\zeta \sim a^{-2}$, yielding a finite dynamical response [44, 64].

It is straightforward to show that the late-time limit of Eq. (3) is

$$\Pi(t) = -3A\omega\zeta\frac{\cos(\omega t - \phi)}{\sqrt{(\omega\tau_\zeta)^2 + 1}}, \qquad (5)$$

where $\zeta = \zeta(a_0)$ and the phase-shift is given by $\tan\phi = \omega\tau_\zeta$. For nearly instantaneous relaxation or very slow drives, $\omega\tau_\zeta \ll 1$, we recover the Navier-Stokes limit $\Pi_{\rm NS}(t) = -3A\omega\zeta\cos(\omega t)$. However, even for moderately slow drives the MIS attractor will never converge to the NS result.

The bulk pressure has units of energy density. Therefore, it is convenient to normalize Π by the equilibrium

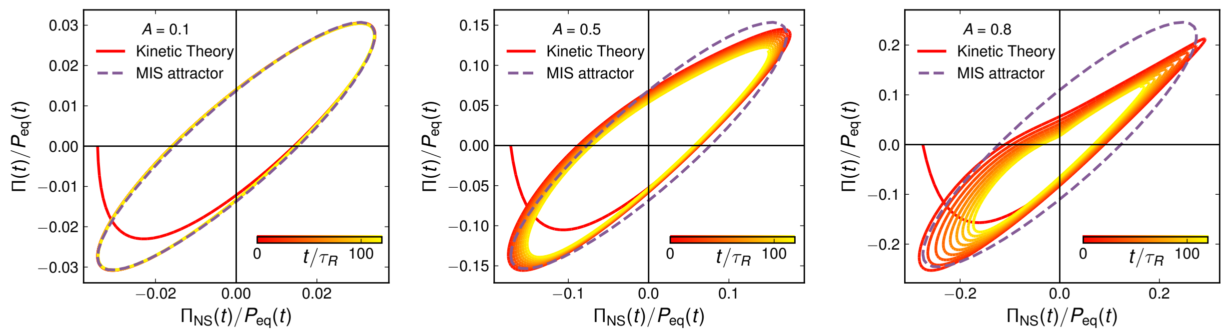

FIG. 2. Massive kinetic theory: attractor solution (limiting cycle) for oscillating expansion at different drive amplitude A. At small amplitude (left panel) the full kinetic theory solution (solid line) agrees with the linear-response MIS attractor. At larger amplitude (center and right panels) kinetic theory predicts a nonlinear attractor that deviates strongly from the MIS attractor. The asymmetric shape arises because also the equation of state evolves with time, cf. Fig. 3.

pressure $P_{\rm eq}$ and by the bulk susceptibility normalized by pressure, $\zeta/(\tau_\zeta P_{\rm eq})$, which are properties of the fluid and universal scaling functions of temperature T/T_F and interaction a_0. Their values from microscopic computations for strongly correlated Fermi gases are reported in Refs. [72, 73]. Finally, the linear response is proportional to the characteristic perturbation amplitude $3A\omega\tau_\zeta$. Therefore, we consider the ratio

$$\frac{\Pi}{P_{\rm eq}}\left(\frac{\zeta}{P_{\rm eq}\tau_\zeta}\right)^{-1}(3A\omega\tau_\zeta)^{-1}=\frac{\Pi}{3A\omega\zeta}, \tag{6}$$

which is the universal dimensionless bulk pressure. We plot the time evolution of $\Pi(t)$ against the expected Navier-Stokes evolution in Fig. 1 for different oscillation frequencies $\omega\tau_\zeta$ and initial values of $\Pi(0)$. We observe that the full solution approaches the late-time limit, Eq. (5), already at time $t/\tau_\zeta = 3$. For slow drives $\omega\tau_\zeta \ll 1$, different initial conditions collapse onto the narrow ellipse along the Navier-Stokes limit (diagonal). The width of the ellipse can be defined as the extent $\Delta\Pi$ of the bulk pressure Π where $\Pi_{\rm NS}=0$, and is given by $\Delta\Pi/(3A\omega\zeta) = 2\omega\tau_\zeta/[1+(\omega\tau_\zeta)^2] \leq 1$. The maximum width is reached when the drive frequency $\omega = \tau_\zeta^{-1}$ equals the relaxation rate; this can serve as an experimental determination of τ_ζ^{-1}. For higher values of the drive frequency, the phase shift ϕ between Navier-Stokes and actual bulk pressure increases, but the width decreases. This is because at large drive frequencies, the system is not able to follow and respond quickly enough.

Massive kinetic theory.—In order to study hydrodynamic attractors in the nonlinear regime of large drive amplitudes, we employ relativistic kinetic theory of massive particles in relaxation time approximation. We generalize Ref. [74] to a time-dependent metric $ds^2 = dt^2 - b(t)^2(dx^2+dy^2+dz^2)$ with scale factor $b(t)$, which is equivalent to the homogeneous and isotropic expansion of gas with rate $\theta = 3\dot{b}/b$. This isotropic expansion is equivalent to the change of scattering length discussed above [64] and a convenient way to compute the bulk viscosity [50]. In spherical coordinates the phase-space distribution function obeys the following Boltzmann equation [75, 76]

$$\left[\partial_t - 2\frac{\dot{b}}{b}p\frac{\partial}{\partial p}\right]f(t,p) = -\frac{f(t,p)-f_{\rm eq}(p,T(t))}{\tau_R}, \tag{7}$$

where τ_R is the relaxation time. We take τ_R as a constant input parameter, which has been computed microscopically for ultracold atoms [53, 72, 77, 78]. The effective temperature T in the equilibrium distribution $f_{\rm eq}(p,T) = e^{-\sqrt{m^2+p^2b^2(t)}/T}$ is defined such that that the instantaneous energy density $e(t) = e_{\rm eq}(T(t))$ matches the equilibrium energy density, where

$$e(t) = \int \frac{d^3p}{(2\pi)^3} b(t)^3\sqrt{m^2+b(t)^2p^2}\, f(t,p), \tag{8}$$

$$e_{\rm eq}(T) = \frac{3T^4}{\pi^2}\left(\frac{m^3}{6T^3}K_1\left(\frac{m}{T}\right)+\frac{m^2}{2T^2}K_2\left(\frac{m}{T}\right)\right). \tag{9}$$

The implicit general solution of Eq. (7) [74, 76, 79] is

$$f(t,p) = e^{-t/\tau_R}f_0(pb^2(t)) + \frac{1}{\tau_R}\int_0^t dt' e^{(t'-t)/\tau_R} f_{\rm eq}\left(p\frac{b^2(t)}{b^2(t')}, T(t')\right), \tag{10}$$

where one has to determine self-consistently the effective temperature evolution $T(t)$. This can be done by substituting the distribution, Eq. (10), into the energy density integral, Eq. (8), and equating it to the equilibrium energy density, Eq. (9). The resulting integral equation can be solved iteratively for $T(t)$ and then inserted into the solution for the distribution, Eq. (10) (see Appendix for details).

We solve Eq. (7) for periodic expansion and contraction described by $b(t) = 1 + A\sin\omega t$. In particular, we compute the time evolution of bulk pressure $\Pi(t) \equiv \frac{1}{3}T^i_{\ i} - P_{\rm eq}$. The Navier-Stokes expectation is the same as in the linear case: $\Pi_{\rm NS}(t) = -3A\omega\zeta\cos(\omega t)$. For massive

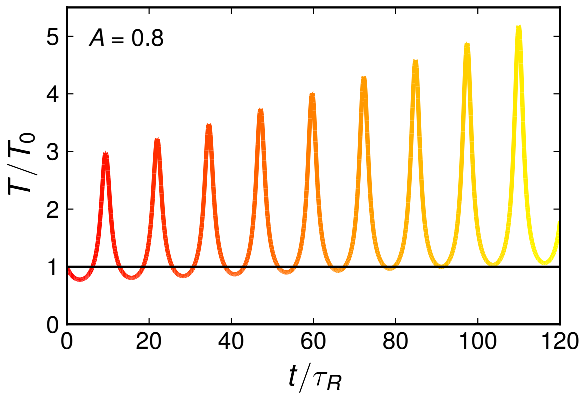

FIG. 3. Temperature evolution for a strong periodic drive. The temperature oscillation has large amplitude, deviates from the sinusoidal shape of the drive, and the peak height grows with each cycle due to dissipative heating. This temperature dependence leads to the asymmetric orbits of the attractor in Fig. 2(c), which do not close after one period.

Boltzmann particles, the ratio of bulk viscosity to pressure $\zeta/(\tau_R P_{\rm eq})$ is a monotonically increasing function of m/T between zero and 2/3 [74]. In Fig. 2 we show the ratio $\Pi(t)/P_{\rm eq}$ for different amplitudes $A = 0.1, 0.5, 0.8$. For concreteness, we consider $m/T = 5$, $\omega\tau_R = 0.5$, and $\zeta/(\tau_R P_{\rm eq}) = 0.23$ at $t = 0$. For small amplitude $A = 0.1$ (left panel), the full kinetic solution approaches the linear-response MIS attractor. For a larger amplitude $A = 0.5$ (central panel), the kinetic theory attractor deviates from MIS attractor. In the right-most panel of Eq. (7) we show the kinetic solution in a strongly nonlinear regime $A = 0.8$. The nonlinear kinetic attractor drifts in time, which we associate with the time-evolving equation of state. In Fig. 3 we show the temperature evolution in the strongly nonlinear regime $A = 0.8$. Due to entropy production, the system is heating up and the average temperature increases with time. This modifies the average $\zeta/(\tau_R P_{\rm eq})$ and $P_{\rm eq}$ values. The cycle averaged energy grows in time quadratically with drive amplitude A as $\langle \dot{e}\rangle \propto \zeta A^2\omega^2$ [64], which is neglected in linear response.

Conclusions.— In conclusion, we have shown how the out-of-equilibrium response to periodic drives realizes a new form of cyclic hydrodynamic attractor beyond the paradigm of monotonic expansion. These attractors are solutions of MIS hydrodynamics or kinetic theory and differ from the Navier-Stokes prediction even at late times and for slow drives. This is a crucial advantage for their experimental identification, where a long-time measurement can yield very precise results. Such measurements are already feasible on the ultracold atom platform, where the nonequilibrium equation of state can be measured with high time resolution [80, 81]. Observing closed cycles of deviations from equilibrium measures, as shown in Fig. 2, would provide clear evidence of attractor behavior. This would manifest not only in the linear regime but also in the nonlinear regime under strong driving conditions. The experimental observation of hydrodynamic attractor phenomena in ultracold atom systems would be a remarkable validation of a theoretical concept originally developed in the study of high-energy nuclear collisions. Such a discovery would further deepen the phenomenology connections between these two research communities [82, 83].

Finally, our work motivates the study of attractors for generic drives beyond the sinusoidal shape. These can be studied in the MIS and kinetic theory formulations presented in this Letter. Moreover, cyclic attractors can also be studied in holographic theories of strongly interacting systems, which is a promising route to understand them from transseries point of view [13].

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Massive kinetic theory

We consider a homogeneous and isotropically expanding system (Hubble expansion) of massive particles in a Friedmann-Lemaître-Robertson-Walker (FLRW) metric (see [76, 84] for Hubble expansion of massless particles and [74] for massive particles in Bjorken expansion)

$$ds^2 = dt^2 - b(t)^2(dx^2 + dy^2 + dz^2). \tag{11}$$

The nonzero Christoffel symbols are $\Gamma^i_{0j} = \delta^i_j \dot{b}/b$, and $\Gamma^0_{ij} = b\dot{b}\delta_{ij}$, where i, j are spatial indices. For momentum integrals, we need the integration measure $\sqrt{-g} = b^3$, where g is the determinant of the metric. We assume that in the coordinates of Eq. (11), the fluid is static, i.e., $u^\mu = (1, 0, 0, 0)$, but the expansion rate $\theta = D_\mu u^\mu = 3\dot{b}/b$ is nonzero, where D_μ is the covariant derivative.

For homogeneous system the single-particle distribution function $f(t, \vec{p})$ obeys the relativistic Boltzmann equation [75, 76]

$$\left[p^\mu\partial_\mu - \Gamma^i_{\alpha\beta}p^\alpha p^\beta\frac{\partial}{\partial p^i}\right]f(t,\vec{p}) = -C[f], \tag{12}$$

where $p^0 = \sqrt{m^2+|\vec{p}|^2b^2}$ and $\vec{p}\equiv(p^1,p^2,p^3)$. For the FLRW metric, Eq. (11), we can simplify the Boltzmann equation to

$$\left[\partial_t - 2\frac{\dot{b}}{b}p\frac{\partial}{\partial p}\right]f(t,p) = -\frac{1}{p^0}C[f]. \tag{13}$$

This can be solved by employing a scaled momentum $\tilde{p} = pb^2(t)/b^2(t_0)$ and defining $F(t,\tilde{p}) = F(t,pb^2(t)/b^2(t_0)) \equiv f(t,p)$. In these coordinates the Boltzmann equation reads

$$\partial_t F(t,\tilde{p})|_{\tilde{p}} = -\frac{1}{p^0}C[F]. \tag{14}$$

For collisionless systems, the solution is $f(t,p) = F(t_0,\tilde{p}) = f(t_0,pb^2(t)/b^2(t_0))$.

We consider kinetic theory in Relaxation Time Approximation (RTA), for which the collision kernel is given by

$$C[f] = \frac{-p^\mu u_\mu\left(f(t,\vec{p}) - f_{\rm eq}(t,\vec{p})\right)}{\tau_R}. \tag{15}$$

For computational simplicity, we will consider classical Boltzmann particles for which the equilibrium distribution is determined by temperature T as

$$f_{\rm eq}(p,T) = e^{-\sqrt{m^2+b^2p^2}/T}. \tag{16}$$

For the collision kernel to satisfy energy conservation, we require that the instantaneous energy densities of $f(t,\vec{p})$ and $f_{\rm eq}(t,\vec{p})$ are equal: $e(t) = e_{\rm eq}(T(t))$, where (degeneracy $\nu = 1$)

$$e(t) = \int\frac{d^3p}{(2\pi)^3}b(t)^3\sqrt{m^2+b(t)^2p^2}f(t,p), \tag{17}$$

$$\begin{aligned} e_{\rm eq}(T) &= \int\frac{d^3p}{(2\pi)^3}b(t)^3\sqrt{m^2+b(t)^2p^2}f_{\rm eq}(p,T) \\ &= \frac{3T^4}{\pi^2}\left(\frac{m^3}{6T^3}K_1\left(\frac{m}{T}\right) + \frac{m^2}{2T^2}K_2\left(\frac{m}{T}\right)\right), \end{aligned} \tag{18}$$

and the time dependence of $T(t)$ has to be determined self-consistently.

For the RTA collision kernel, the Boltzmann equation Eq. (14) can be written as

$$\partial_t F(t,\tilde{p}) = \frac{F_{\rm eq}(T(t),\tilde{p}) - F(t,\tilde{p})}{\tau_R}, \tag{19}$$

which has the solution (we take $t_0 = 0$ and $b(t_0) = 1$)

$$F(t,\tilde{p}) = F(0,\tilde{p})e^{-t/\tau_R} + \frac{1}{\tau_R}\int_0^t dt' e^{-(t-t')/\tau_R}F_{\rm eq}(\tilde{p},T(t')) \tag{20}$$

or, alternatively (cf. [76]),

$$\begin{aligned} f(t,p) &= e^{-t/\tau_R}f(0,pb^2(t)) \\ &+ \frac{1}{\tau_R}\int_0^t dt' e^{(t'-t)/\tau_R}f_{\rm eq}\left(p\frac{b^2(t)}{b^2(t')},T(t')\right). \end{aligned} \tag{21}$$

We take the initial distribution to be in equilibrium, i.e., $f(0,p) = f_{\rm eq}(p,T_0)$. Then

$$\begin{aligned} f(t,p) &= e^{-\sqrt{m^2+p^2b^4(t)}/T_0}e^{-t/\tau_R} \\ &+ \frac{1}{\tau_R}\int_0^t dt' e^{-(t-t')/\tau_R}e^{-\sqrt{m^2+p^2b^4(t)/b^2(t')}/T(t')} \end{aligned} \tag{22}$$

In order to solve Eq. (22) we need to determine the instantaneous equilibrium temperature $T(t)$. Performing the energy density integral of Eq. (22) we obtain

$$\begin{aligned} &\frac{3T(t)^4}{\pi^2}\left(\frac{m^3}{6T(t)^3}K_1\left(\frac{m}{T(t)}\right) + \frac{m^2}{2T(t)^2}K_2\left(\frac{m}{T(t)}\right)\right) \\ &= T_0^4 I\left(\frac{m}{T_0},b(t)\right)e^{-t/\tau_R} \\ &+ \frac{1}{\tau_R}\int_0^t dt' e^{-(t-t')/\tau_R}T^4(t')I\left(\frac{m}{T(t')},\frac{b(t)}{b(t')}\right), \end{aligned} \tag{23}$$

where we used that $e(t) = e_{\rm eq}(T(t))$ and defined a dimensionless integral

$$I(y,z) = \int_0^\infty\frac{4\pi x^2dx}{(2\pi)^3}\sqrt{y^2+x^2}e^{-\sqrt{y^2+z^2x^2}}. \tag{24}$$

For each new time step $t = n\delta t$, we take an initial guess of $T(n\delta t) = T((n-1)\delta t)$. We use cubic spline interpolation of $T(t)$ in the integrals on the right-hand side of Eq. (23) to compute the energy density. We then perform a root-finding algorithm to determine an updated value for $T(n\delta t)$. We perform this procedure until the relative temperature change falls below 10^{-6} and then proceed to the next time step. The python implementation of the solver converges in a few minutes for typical settings: $t_{\rm max} = 60$, $N_t = 1000$, $\tau_R = 0.5$, $m = 5$, $\omega = 1$, $T_0 = 1$, $A = 0.5$. We show the temperature evolution for $A = 0.1, 0.5, 0.8$ in Fig. 4. For small amplitudes, the temperature is oscillating symmetrically around the equilibrium value. For larger amplitude values the oscillation is asymmetric and deviates from the sinusoidal drive. We also observe a drift in average temperature, which is due to entropy production during the cycle.

In the main text of the paper, Fig. 2, we discuss the time evolution of the bulk pressure. In kinetic theory we define the bulk pressure as

$$\begin{aligned} \Pi(t) &= \frac{1}{3}\int\frac{d^3p}{(2\pi)^3}b^3\frac{p^2b^2(t)}{\sqrt{m^2+p^2b^2(t)}}\left(f(t,p) - f_{\rm eq}(t,p)\right) \\ &= \frac{1}{3}R_{\rm eq}(T(t)) - \frac{1}{3}\int\frac{d^3p}{(2\pi)^3}b^3\frac{m^2}{\sqrt{m^2+p^2b^2(t)}}f(t,p) \end{aligned} \tag{25}$$

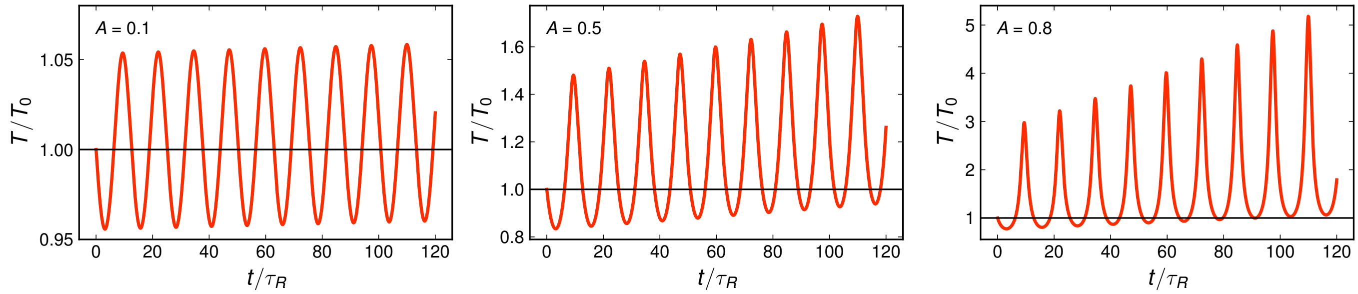

FIG. 4. Temperature evolution for different periodic drives in massive kinetic theory. For larger drive amplitude A, the temperature oscillations grow in amplitude, they become less sinusoidal, and the peak height grows from one cycle to the next.

where we used integration by parts and that $e(t) = e_{\rm eq}(T(t))$. In addition, we defined the trace of the energy-momentum tensor (conformal scale breaking term) as

$$\begin{aligned} R_{\rm eq}(T) &= e - 3P_{\rm eq} \\ &= \int \frac{d^3p}{(2\pi)^3} b(t)^3 \frac{m^2}{\sqrt{m^2 + b(t)^2 p^2}} f_{\rm eq}(p,T) \\ &= \frac{3T^4}{\pi^2}\left(\frac{m^3}{6T^3} K_1\left(\frac{m}{T}\right)\right). \end{aligned} \tag{26}$$

Inserting the solution Eq. (22) into Eq. (25) we obtain

$$\begin{aligned} \Pi(t) &= \frac{1}{3} R_{\rm eq}(T(t)) - T_0^4 J\left(\frac{m}{T_0}, b(t)\right) e^{-t/\tau_R} \\ &- \frac{1}{\tau_R}\int_0^t dt' e^{-(t-t')/\tau_R} T^4(t') J\left(\frac{m}{T(t')}, \frac{b(t)}{b(t')}\right), \end{aligned} \tag{27}$$

in terms of the integral

$$J(y,z) = \frac{1}{3}\int_0^\infty \frac{4\pi x^2 dx}{(2\pi)^3} \frac{y^2}{\sqrt{y^2+x^2}} e^{-\sqrt{y^2+z^2x^2}}. \tag{28}$$

For completeness, we give the analytical form of the bulk susceptibility [74] as a function of $z = m/T$ and plot it in Fig. 5: the analytical form reads

$$\frac{\zeta}{P_{\rm eq}\tau_R} = \frac{z^3}{9}\left(-\frac{K_2(z)}{3K_3(z) + zK_2(z)} + \frac{K_1(z)}{K_2(z)} - \frac{K_{i,1}(z)}{K_2(z)}\right) \tag{29}$$

where

$$K_{i,1}(z) = \frac{\pi}{2}[1 - zK_0(z)L_{-1}(z) - zK_1(z)L_0(z)] \tag{30}$$

and K and L are modified Bessel and Struve functions, respectively.

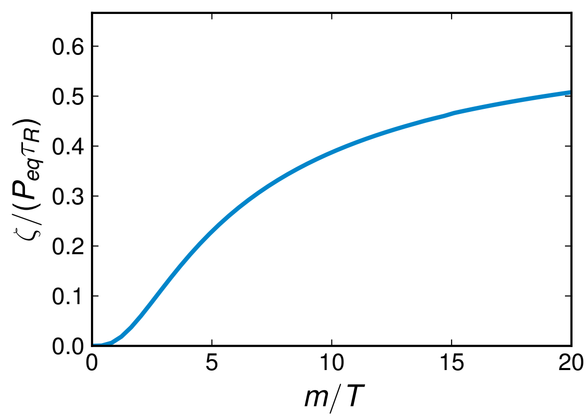

FIG. 5. The bulk susceptibility $\zeta/P_{\rm eq}\tau_R$ in massive kinetic theory as a function of m/T.

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