Quantum Field Theory 1 – Happy Holidays

Lectures: Astrid Eichhorn eichhorn@thphys.uni-heidelberg.de
Tutorials: Zois Gyftopoulos gyftopoulos@thphys.uni-heidelberg.de
Institut für Theoretische Physik, Uni Heidelberg date: 25 December 2025

Name: **matriculation number:**

Practice group:

Exercise	P 1	P 2	P 3	Σ
max. points	30	35	35	100
points				

You should solve only 3 out of the 4 problems. Problems 1 and 4 are compulsory. You are free to choose either Problem 2 or Problem 3 as your third exercise.
Don't forget to write your name and your practice group on all the sheets you hand in.

Good luck!

Problem 1: Comprehension questions (30 points)

Provide answers to the following conceptual questions. Your answers should be short but provide enough details for a comprehensive answer. You should use equations to provide context in your answers but you don't have to do any calculations.

i) Why are antiparticles necessary for a relativistic quantum theory of charged particles? What types of fields are used to describe such particles/antiparticles?

ii) Explain briefly how maintaining symmetry under local transformations requires the introduction of additional fields. Provide also an example of such a theory and explain what we mean by "gauge symmetry".

iii) What is the S-matrix and why does it have to be unitary? Briefly outline the steps for calculating the S-matrix using the LSZ formalism and the path-integral or canonical quantization.

Problem 2A: Complex fields and their charges (35 points)

Consider the Lagrangian of a complex scalar field $\mathcal{L} = \int d^4x \left[(\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi\right]$.

a) (4 points) Show that the Lagrangian is invariant under global $U(1)$ transformations

$$\Phi \to e^{ia}\Phi \ . \tag{1}$$

b) (4 points) Show that the associated Noether current is

$$j^\mu = i(\partial^\mu \Phi)^\dagger \Phi - i\Phi^\dagger \partial^\mu \Phi \ , \tag{2}$$

c) (6 points) Calculate the equations of motions for the fields $\Phi^\dagger$ and Φ, and show that the associated Noether current j^μ is conserved, i.e. that $\partial_\mu j^\mu = 0$.

d) (12 points) The field operators in terms of creation and annihilation operators in the Schrödinger picture read

$$\Phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left\{a_{\boldsymbol{p}}\, e^{-i\boldsymbol{px}} + b^\dagger_{\boldsymbol{p}}\, e^{i\boldsymbol{px}}\right\} , \tag{3}$$

with $\omega_{\boldsymbol{p}}^2 = \boldsymbol{p}^2 + m^2$, and $b, b^\dagger$ and $a, a^\dagger$ obey the commutation relations

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^{(3)}(p-q) = \left[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] , \quad \text{and all others vanish.} \tag{4}$$

Show that charge operator $Q = \int d^4x j^0$ associated to the global $U(1)$ symmetry in terms of creation and annihilation operators (ignoring all infinite constant terms) takes the form:

$$Q = \int \frac{d^3p}{(2\pi)^3} \left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}}\right) \ . \tag{5}$$

e) (6 points) Act with the charge operator (5) on the following 1-particle states to calculate their charges.

$$|0\rangle \quad , \quad a^\dagger_{\boldsymbol{p}} |0\rangle \quad , \quad b^\dagger_{\boldsymbol{p}} |0\rangle \quad , \quad b^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} |0\rangle \quad , \quad a^\dagger_{\boldsymbol{p}_1} \dots a^\dagger_{\boldsymbol{p}_n} |0\rangle \tag{6}$$

f) (3 points) Why do we need operators $b, b^\dagger$ in addition to $a, a^\dagger$ and what is the physical interpretation of b and $b^\dagger$?

Problem 2B: Quantization of commuting spinors (35 points)

Suppose we start from a theory described by the Dirac Lagrangian $\mathcal{L} = \bar{\Psi}\,(i\gamma^\mu\partial_\mu - m)\,\Psi$, where Ψ is a Dirac spinor.

a) (4 points) Calculate the conjugate momentum of the spinor field Ψ and its hermitian conjugate $\Psi^\dagger$.

b) (12 points) Assume now the following mode expansion for the Dirac spinor

$$\Psi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \sum_s \left\{ a^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-i\boldsymbol{px}} + b^{\dagger,s}_{\boldsymbol{p}} \bar{v}^s(\boldsymbol{p}) e^{+i\boldsymbol{px}} \right\} \ , \tag{7}$$

with the following commutation relations for the creation/annihilation operators:

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right] = (2\pi)^3 \delta^{(3)}(p-q) = \left[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{p}}\right] \ , \ \text{ and all the others vanishing.} \tag{8}$$

Show that the Hamiltonian takes the form

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p \sum_s \left\{ a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} + \text{const. terms} \right\} \ . \tag{9}$$

c) (8 points) Calculate the energy of the following 1-particles states

$$|0\rangle \quad , \quad a^{\dagger,s}_{\boldsymbol{p}} |0\rangle \quad , \quad b^{\dagger,s}_{\boldsymbol{p}} |0\rangle \quad , \quad a^{\dagger,s}_{\boldsymbol{p}_1} b^{\dagger,s'}_{\boldsymbol{p}_2} |0\rangle \quad , \quad b^{\dagger,s_1}_{\boldsymbol{p}_1} \dots b^{\dagger,s_n}_{\boldsymbol{p}_n} |0\rangle \ . \tag{10}$$

What is the problem with this Hamiltonian?

d) (4 points) Another equivalent version of the Dirac Lagrangian is the following:

$$\mathcal{L} = -\frac{i}{2}(\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \frac{i}{2}\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \equiv \frac{i}{2}\bar{\Psi}\left(i\gamma^\mu\overleftrightarrow{\partial}_\mu - m\right)\Psi \ . \tag{11}$$

The usual Dirac Lagrangian is equivalent up to a total derivative. Calculate the conjugate momenta of Ψ and $\Psi^\dagger$ from the new Lagrangian (11).

e) (4 points) Calculate the new Hamiltonian using the Lagrangian (11). Does it agree with Eq. (9)? Is it equivalent with the previous Hamiltonian?

f) (3 points) Why do we need operators $b, b^\dagger$ in addition to $a, a^\dagger$ and what is the physical interpretation of b and $b^\dagger$?

Problem 3: 2-to-2 scattering (35 points)

Suppose you have a scalar ϕ^4 theory described by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \tag{12}$$

One can compute the time-ordered n-point functions via the master formula

$$\langle 0|\mathcal{T}\left[\phi(x_1)\dots\phi(x_n)\right]|0\rangle = \frac{\langle 0_{\text{free}}|\phi_I(x_1)\dots\phi_I(x_n)\exp\left(i\int d^4x\mathcal{L}_{\text{int}}\right)|0_{\text{free}}\rangle}{\langle 0_{\text{free}}|\exp\left(i\int d^4x\mathcal{L}_{\text{int}}\right)|0_{\text{free}}\rangle} \ , \tag{13}$$

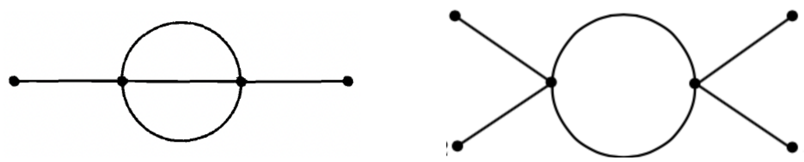
where ϕ_I denote fields in the interaction picture.

a) (14 points) Write down the Wick contractions for the numerator of Eq. (13) for the 3 and 4- point functions

$$\langle 0|\mathcal{T}\left[\phi(x_1)\phi(x_2)\right]|0\rangle \quad , \quad \langle 0|\mathcal{T}\left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right]|0\rangle \ , \tag{14}$$

up to **second order** in the coupling λ respectively (**ignore contractions with disconnected parts and do not write the zeroth order in** λ), and draw the corresponding Feynman diagrams. These higher order diagrams correspond to quantum corrections and are always **loop** diagrams.

b) (4 points) Calculate the symmetry factors corresponding to the following two of the above diagrams:



c) (2 points) What happens to the 3- point function $\langle 0|\mathcal{T}\left[\phi(x_1)\phi(x_2)\phi(x_3)\right]|0\rangle$?

d) (3 points) Substitute now the interaction term from $\frac{\lambda}{4!}\phi^4$ to $\frac{\kappa}{3!}\phi^3$. What happens to the 2, 3 and 4- point functions to first order in κ?

e) (8 points) Write down the Wick contractions (ignore contractions with disconnected parts) for the 3- point function at κ^3- order which are fully connected and draw the corresponding diagrams.

f) (4 points) Explain how the same expressions could be derived using the path-integral formalism (there is no need to provide whole re-derivation of the above).

One naturally could adapt this problem with writing down Feynman rules and the calculation of some of the diagrams.