Quantum Field Theory II

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This is a first draft of the script to Prof. Wetterich's QFT II lecture at the university of Heidelberg in the summer semester 2006. The latest version can be found at the course homepage `http://www.thphys.uni-heidelberg.de/~cosmo/view/Main/QFTWetterich`.

If you find an error, please check whether you have the newest version (see date above), and if so report it to `thequantumfive@gmx.net`. Also if you find something confusing and think it should be explained in more detail or more clearly, please let us know. Your feedback will be the main force to improve the script.

The script was LaTeXed by the students:

Contents

1 Yukawa Model

1.1 Fermion-Scalar Coupling

Fermion-scalar coupling was originally introduced to describe strong interaction. So we will also study it in this context.

1.1.1 Meson Exchange in Strong Interactions

Scalar Field Theory (Repetition)

$$Z = \int \mathcal{D}\varphi \, e^{iS[\varphi]} \tag{1.1}$$

$$\langle O[\varphi] \rangle = Z^{-1} \int \mathcal{D}\varphi \, O[\varphi] e^{iS[\varphi]} \tag{1.2}$$

E.g. for $2 \to 2$ scattering you need the 4-point function $\varphi^*(p_1)\varphi(-p_2)\phi(p_3)\phi^*(-p_4)$.

`include figure 1`

Our theory will have no scalar-scalar or fermion-fermion interaction, only 2 fermion - 1 scalar interaction.

We consider one complex scalar field φ that describes both π^+ and π^-. We associate φ with π^+ and φ^* with π^-. In the free theory we have the action

$$S_\varphi = -\int d^4x \, \left\{ \partial^\mu \varphi^* \partial_\mu \varphi + m^2 \varphi^* \varphi \right\}. \tag{1.3}$$

Charged Pion Exchange

The exchange of the charged pions π^+ and π^- contributes to proton-neutron scattering.

`include figures 2 and 3`

It is described by a so called **Yukawa interaction**, i.e. 2 fermion - 1 scalar interaction. Because the proton and neutron are not elementary particles, the Yukawa model has only a limited range of application. It only describes the limit of the nuclei having small momenta (less than 100 MeV) and large separation (probably several Fermi). At higher energies gluons (self-interacting spin 1 particles) are interchanged and you need QCD.

The free fermion part of the action is

$$S_\psi = -\int d^4x \, \{ \bar{p} i \gamma^\mu \partial^\mu p + \bar{n} i \gamma^\mu \partial^\mu n + i m_p \bar{p} p + i m_p \bar{p} p \}. \tag{1.4}$$

The total action is given by the free boson term S_φ, the free fermion term S_ψ, and a Yukawa interaction term S_Y.

Yukawa Interaction

The Yukawa interaction is given by

$$S_\psi = -ig \int d^4x \, \{\bar{p}n\varphi + \bar{n}p\varphi^*\}\,, \tag{1.5}$$

where g is a dimensionless coupling constant. It is a charge exchange interaction.
`include figures 4 and 5`

$$\langle pn\bar{p}\bar{n}\rangle = Z^{-1} \int \mathcal{D}\varphi \int \mathcal{D}\psi \, pn\bar{p}\bar{n}e^{iS}, \tag{1.6}$$

where $\int \mathcal{D}\psi$ means integration over the p and n fields.

Isospin Symmetry

The proton and neutron form a doublet $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$ which is invariant under $SU(2)$ transformations,

$$\psi \quad \rightarrow \quad e^{i\frac{\alpha}{2}\tau}\psi, \tag{1.7}$$

where τ are the Pauli spin matrices and $\alpha \in \mathbb{R}^3$. This symmetry has nothing to do with space rotations, i.e. spin. The components p and n of ψ both have spin, they are both 4-component Dirac spinors, i.e. ψ has 8 components.

This is an **internal symmetry**. It acts on an **internal quantum number**, called **isospin**.

The proton is ψ with isospin up. The neutron is ψ with isospin down. The three component of the isospin is related to the charge by the formula

$$Q = \frac{1}{2} + I_3, \tag{1.8}$$

because the proton has $I_3 = +\frac{1}{2}$, $Q = +1$ and the neutron has $I_3 = -\frac{1}{2}$, $Q = 0$. But the full symmetry is also around the I_1 and I_2 axis.

Isospin is a very good symmetry for strong interactions. We have

$$\bar{\psi} = \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix} = \quad \rightarrow \quad e^{-i\frac{\alpha}{2}\tau}\bar{\psi}, \tag{1.9}$$

$$\bar{p}p + \bar{n}n \quad \rightarrow \quad \bar{\psi}(i\gamma^\mu\partial_\mu + im)\psi + \bar{\psi}i(m_p - m_n)\tau_3\psi. \tag{1.10}$$

The second term cancels when the proton and neutron mass are equal. This is the limit of isospin symmetry.

$$m_p = 938.272\,\text{MeV}, \quad m_n = 939.565\,\text{MeV}, \tag{1.11}$$

$$m_n - m_p = 1.293\,\text{MeV}, \quad \frac{m_n - m_p}{m_p} = 0.138\,\%. \tag{1.12}$$

The pions must also have isospin, or the Yukawa interaction will not be invariant. With a little representation theory we find.

`I didn't understand it. The reasoning was that` $2 \times 2 = 3 + 1$, `but` $2 \times 2 \neq 2 + 2$.

$$\pi = (\pi_1, \pi_2, \pi_3), \tag{1.13}$$
$$\pi^0 = \pi_3 \tag{1.14}$$
$$\pi^+ = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2) \tag{1.15}$$
$$\pi^- = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2) \tag{1.16}$$
$$\tag{1.17}$$

To see that the Yukawa interaction is isospin invariant we write it as

$$S_Y = -i\frac{h}{2}\int d^4x\, \bar{\psi}\tau\pi\psi, \tag{1.18}$$

the factor of $\frac{h}{2}$ is just a convention.

`Something about a triplet state I didn't understand.`

$$S_Y = -i\frac{h}{2}\int d^4x \left\{\bar{p}p\pi^0 - \bar{n}n\pi^0 + \sqrt{2}(\bar{p}n\pi^+ + \bar{n}p\pi^-)\right\} \tag{1.19}$$

So the exchange of charged pions is just a part of all pion exchange. We will restrict our discussion to the charged pions ($g = \frac{h}{\sqrt{2}}$).

1.1.2 Feshbach Resonance for Atoms

We are talking about the scattering of fermionic atoms (e.g. Li-7 or K) through bound state exchange. The two atom state is described by $\psi = \begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow\end{pmatrix}$. The molecule, i.e. the bound state of the two atoms, is described by φ. The action is

$$S = -\int dt\, d^3x \left\{-i\psi^\dagger\partial_t\psi + \frac{1}{2M}\nabla\psi^\dagger\nabla\psi - i\varphi^*\partial_t\varphi + \frac{1}{4M}\nabla\varphi^*\nabla\varphi + \nu(B)\varphi^*\varphi - h(\psi_\uparrow\psi_\downarrow\varphi^* - \psi_\uparrow^*\psi_\downarrow^*\varphi)\right\} \tag{1.20}$$

This describes the interactions

`include figures 6 and 7`

The symmetry is $U(1)$ (phase transformations):

$$\psi \to e^{i\alpha}\psi, \quad \varphi \to e^{2i\alpha}\varphi \tag{1.21}$$