QFT II - Problem Set 1

(27) Spin Sums

Evaluating matrix elements of Fermions is greatly simplified by so called spin sums. In addition, spin sums are often what one is really interested in: either because an experiment has no control over in and outgoing spins, or because some process in nature involves all possible spins and one is only interested in the total amplitude. In this case, one averages over incoming and sums over outgoing spins.

You will derive the spin sums here. Please note that depending on conventions, the relations will change. For convenience, we will give the results in the conventions of Peskin & Schroeder at the end of this exercise. Yet, in the conventions of this lecture, the Dirac equation is

$$(\gamma^\mu \partial_\mu + m)\Psi(x) = 0,$$

which is solved by the positive frequency Ansatz

$$\Psi(x) = u^s(p) \exp(ip_\mu x^\mu),$$

as well the negative frequency solutions

$$\Psi(x) = v^s(p) \exp(-ip_\mu x^\mu),$$

where $p_0 = -E$. At rest, $p_\mu = (-m, 0, 0, 0)$ and we will use the “slash” notation $\not{a} = a_\mu \gamma^\mu$.

a) Using $\Psi = u^s(p)e^{ipx}$, what is the Dirac equation for $u^s(p)$?

b1) At rest, $p_\mu = (-m, 0, 0, 0)$. Show that in this case $u^s(p_{rest}) = \sqrt{m}\begin{pmatrix} \chi_s \\ \chi_s \end{pmatrix}$ with $\chi_{1/2} = (1,0)^T$, $\chi_{-1/2} = (0,1)^T$ solves the Dirac equation for u.

b2) What is the solution for $v^s(p_{rest})$?

To prove the spin sum relation, consider the Ansatz $\sum_s u^s(p)\bar{u}^s = A$, or alternatively, with explicit spinor indices: $\sum_s u^s_\alpha \bar{u}^s_\beta = A_{\alpha\beta}$, where A is a $4x4$ matrix.

c1) Use this Ansatz to get an expression for $\sum_s u^s(p)\bar{u}^s(p)$ up to a proportionality constant C. *Hints: act with* $(i\not{p} + m)$ *on* $\sum_s u^s(p)\bar{u}^s(p) = A$. *Recall that* $\not{p}^2 = p^2 = -m^2$ *to find* A.

c2) Fix C by equating the expression you just got for the spin sum and $\sum_s u^s\bar{u}^s$ for a particle at rest. This yields the final expression for the spin sum.

d) Derive a similar expression for $\sum_s v^s(p)\bar{v}^s(p)$.

Just for comparison: in the conventions of Peskin & Schroeder, the spin sum formulae are

$$\sum_s u^s_a \bar{u}^s_b = (\not{p} + m)_{ab} \qquad \sum_s v^s_a \bar{v}^s_b = (\not{p} - m)_{ab}$$

(28) Electron-Muon scattering

Consider the scattering of electrons and muons. Let us choose the momenta and spins as follows: the incoming electron has $(momentum, spin) = (p, s)$, the outgoing electron (p', s'), the incoming muon has (k, r) and the outgoing muon (k', r').

a) Draw the (tree-level) Feynman diagram for Electron-Muon scattering.

b) Using the following Feynman rules, get an expression for the scattering Amplitude $\mathcal{M}$. *Hint: for convenience, perform the contraction* $g_{\mu\nu}\gamma^\nu \to \gamma_\mu$

Feynman rules to get $\mathcal{M}$:

(i) Vertex: $-ie\gamma^\mu$

(ii) Photon propagator for momentum q: $-ig_{\mu\nu}/q^2$

(iii) Incoming Fermions u, outgoing Fermions $\bar{u}$

(iv) Incoming Antifermions $\bar{v}$, outgoing Antifermions v

(v) Multiply the entire diagram by a factor of i

c) Remember that for a bilinear involving γ^μ, the relation $(\bar{\Psi}\gamma^\mu\chi)^* = \bar{\chi}\gamma^\mu\Psi$ holds. Using this, write down $\mathcal{M}^*$. *Hint: if you used γ^μ and γ_μ for $\mathcal{M}$, use γ^ν and γ_ν for $\mathcal{M}^*$.*

d) Compute $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$ and group electrons and muons together, i.e. $|\mathcal{M}|^2 = factors \times [u's \text{ of electrons}] \times [u's \text{ of muons} \ldots]$.

e) Suppose that our experiment is blind to spins. We therefore average over ingoing spins (s, k) and sum over outgoing spins (s', k'), i.e. we want to compute

$$\frac{1}{4}\sum_{s,k,s',k'} |\mathcal{M}(s,k,s',k')|^2.$$

e1) Put in the explicit spinor indices, i.e. instead of $\bar{u}^{s'}(p')\gamma^\mu u^s(p)\ldots$ write $\bar{u}^{s'}_a(p')\gamma^\mu_{ab}u^s_b(p)\ldots$

e2) For the spin sums, we need expressions like $u^s\bar{u}^s$. Rearrange the $u's$ to get this structure. *Hint: with the explicit spinor indices, you can easily do it, because you simply re-arrange c-numbers.*

e3) Use the spin sum formula to replace $u_a\bar{u}_b$ etc.

e4) Re-write the expression you got for the $u's$ as a trace. *So if you'd get an expression like $A_{da}\gamma^\mu_{ab}B_{bd}$ that would be* $\mathrm{Tr}[A\gamma^\mu B]$. *You should get two traces, one involving electrons, one involving muons.*

f) Simplify the expression for $|\mathcal{M}|^2$ by using "Trace-technology". You'll need the relations

$$\mathrm{Tr}(\mathbf{1}) = 4 \tag{1}$$

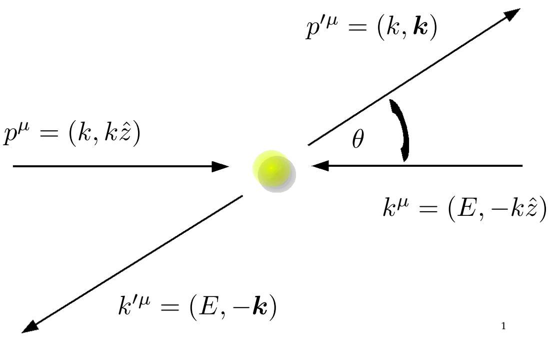
$$\mathrm{Tr}(\text{any odd \# of } \gamma\text{'s}) = 0 \tag{2}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{3}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{4}$$

$$g_{\mu\nu}g^{\mu\nu} = 4 \tag{5}$$

g) We'll neglect the electron mass in the following: $m_e \to 0$. Work out the kinematic to replace (kp), $(p'k), \ldots$ by the energy of the incoming electron and muon and the scattering angle θ. *Hint: orient the incoming particles along the $z-$axis as shown in this figure.*



h) Use the following formula for the cross section in the center of mass frame valid for two particles in the final state where one of them is massless (we will derive this expression for σ in the next problem set):

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\mathcal{M}|^2}{64\pi^2(E+k)^2}.$$

i) What is cross section in the high-energy limit $m_\mu \to 0$?