QFT I - Problem Set 7

(13) Grassmann Variables

Let θ_i be Grassmann variables with $\{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0$ with $i, j = 1, 2, \ldots, n$.

a) Calculate $(\theta_1\theta_2 + \theta_3\theta_4)^2$.

b) What are the solutions x for $x^2 = 1 + \theta_1\theta_2$? Are there solutions for $x^2 = \theta_1\theta_2$?

c) Show that

$$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0\,, \quad \left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta_{ij}\,, \quad \int d\theta_i f(\theta_i) = \frac{\partial}{\partial\theta_i} f(\theta_i)\,.$$

Show that the Dirac δ-distribution can be written as $\delta(\theta) = \theta$.

d) Show that

$$\int \prod_{k=1}^{n} d\theta_k d\theta_k^* \exp\Big\{\sum_{i,j}^{n} \theta_i^* A_{ij}\theta_j + \sum_{i}^{n}(\eta_i^*\theta_i + \theta_i^*\eta_i)\Big\} = \det A \;\exp\Big(-\sum_{i,j}^{n} \eta_i^* A_{ij}^{-1}\eta_j\Big)\,,$$

where the η_i, $i = 1, \ldots, n$ are Grassmann variables too.

(14) Working with the Functional Integral

Consider the action with a complex interacting scalar field

$$\begin{aligned} S[\phi^*, \phi] &= S_0[\phi^*, \phi] + S_I[\phi^*, \phi] \\ &= \int dt d^3x \Big\{\phi^*(t, \boldsymbol{x})\big(i\partial_t - \frac{\triangle}{2M}\big)\phi(t, \boldsymbol{x}) + \frac{\lambda}{2}\big(\phi^*(t, \boldsymbol{x})\phi(t, \boldsymbol{x})\big)^2\Big\} \end{aligned}$$

where S_0 stands for the first term and S_I for the second. The partition function is defined as

$$Z[j^*, j] = \int \mathcal{D}(\phi^*, \phi) \exp\Big\{iS[\phi^*, \phi] - i\int (j\phi^* + j^*\phi)\Big\}.$$

(a) Show

$$Z[j^*, j] = \exp\big[iS_I[i\frac{\delta}{\delta j}, i\frac{\delta}{\delta j^*}]\big]\, Z_0[j^*, j]$$

with

$$Z_0[j^*, j] = \int \mathcal{D}(\phi^*, \phi) \exp\Big\{iS_0[\phi^*, \phi] - i\int (j\phi^* + j^*\phi)\Big\}\,.$$

Hint: Understand $\exp\big[iS_I[i\frac{\delta}{\delta j}, i\frac{\delta}{\delta j^*}]\big]$ *as operation on* $Z_0[j^*, j]$.

(b) Perform a Fourier transform for S_0 and the source terms. Then, evaluate Z_0 explicitly by completion of the square.

(c) Compute the free two-point function

$$\langle\phi^*(p_1)\phi(p_2)\rangle = \Big(i\frac{\delta}{\delta j(p_1)}\Big)\Big(i\frac{\delta}{\delta j^*(p_2)}\Big) Z_0[j^*, j]\Big|_{j=j^*=0}\,,$$

with $p_i = (\omega, \boldsymbol{p})_i$ and the analogously defined four-point function

$$\langle\phi^*(p_1)\phi(p_2)\phi^*(p_3)\phi(p_4)\rangle\,,$$

in terms of the free propagator $G_0(q, p) = \frac{i}{\omega - E(p)}\delta(q - p)$, where $E(p) = \frac{p^2}{2M}$.