Lecture 1

Standard Perturbation Theory for LSS

Ref.: Bernardeau et al Phys. Rep 367, (2002) 1

∘ Bernardeau 1311.2724 (Les Houches)

We will consider scales $\ll \frac{1}{H_0} = 3000 \frac{Mpc}{h}$, or, in Fourier space $k/H_0 \gg 1$, so GR corrections are generically at the ~~sap~~ subpercent level:

~~$k \gtrsim 0.05$~~ $k \gtrsim 0.05\, h/Mpc$ $\frac{k}{H_0} \gtrsim 150$, GR effects are typically $\mathcal{O}\left(\left(\frac{H_0}{k}\right)^2\right) \simeq \mathcal{O}(10^{-5})$.

↳ Newtonian treatment!

[However GR treatment can be necessary in ~~to~~ certain situations, such as MGR with non-linear screening mech.]

The system we describe is a collection of non-relativistic particles in Newtonian approximation, interacting only through gravity:

Single particle action $S = -m\int ds = -m\int a\sqrt{1-v^2}(1+\phi)\,d\tau$

$$\underset{\phi,\, v^2 \ll 1}{\simeq} -m\int a\left(1-\tfrac{1}{2}v^2+\phi\right)d\tau$$

$\vec{v} = \dfrac{d\vec{x}}{d\tau}$ peculiar velocity

$\vec{x}$: comoving coordinate

τ: conformal time

$$\Big[\underbrace{\vec{V}}_{\frac{dR}{dt}} = H\vec{R} + \underbrace{\vec{v}}_{a\dot{\vec{x}}}\Big]$$

$\mathcal{L} = -ma\left(1-\frac{1}{2}v^2+\phi\right)$ $\quad P_i = \dfrac{\partial\mathcal{L}}{\partial \dot{x}^i} = ma\dot{x}^i$ $\quad H = P_i\dot{x}^i - \mathcal{L}$

Eq of motion:

$$\boxed{\begin{aligned} \dot{x}^i &= \frac{P_i}{am} \quad \left(= \frac{\partial H}{\partial P_i}\right) \\ \dot{P}_i &= -am\,\nabla_i\phi \quad \left(= -\frac{\partial H}{\partial x^i}\right) \\ \nabla^2\phi &= \frac{3}{2}\Omega_m \mathcal{H}^2 \delta \end{aligned}}$$

Poisson eq

→ Newtonian limit of (00) Einst. eq

$$\delta \equiv \frac{\rho}{\bar{\rho}} - 1$$

$\rho(\vec{x},\tau) = m\sum_n \delta_D(\vec{x} - \vec{x}_n(\tau))$ $\qquad \bar{\rho} = \dfrac{1}{V}\int d^3\vec{x}\,\rho(\vec{x},\tau) = \dfrac{N}{V}m$

Distribution function: $f(\vec{x},\vec{p},\tau) = \sum_n^N \delta_D(\vec{x}-\vec{x}_n(\tau))\,\delta_D(\vec{p}-\vec{p}_n(\tau))$

- Take the limit $N \to \infty$, $mN \to \text{const}$ (infinite resolution N-body sim.)

- $(V)^{1/3} >>$ any scale we are interested in.

- Neglect 2-body interactions

+ Liouville th.

$$\rightarrow \frac{d}{d\tau} f(\vec{x},\vec{p},\tau) = 0 = \left(\frac{\partial}{\partial \tau} + \dot{x}^i \frac{\partial}{\partial x^i} + \dot{p}_i \frac{\partial}{\partial p_i}\right) f = 0$$

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial \tau} + \frac{p_i}{am}\frac{\partial}{\partial x^i} - am\nabla^i\phi\right) f(\vec{x},\vec{p},\tau) = 0}$$

Vlasov equation

$$\left[\nabla^2\phi = \frac{3}{2}\mathcal{H}^2\Omega_m\delta\right]$$

Vlasov - Poisson system

Moments: $n(\vec{x},\tau) = \int d^3p\, f(\vec{x},\vec{p},\tau) = \bar{n}\,(1+\delta)$ number density

$$v^i(\vec{x},\tau) = \frac{1}{n(\vec{x},\tau)} \int d^3\vec{p}\, \frac{p_i}{a m}\, f(\vec{x},\vec{p},\tau) \quad \text{velocity}$$

ocity → $\sigma^{ij}(\vec{x},\tau) = \frac{1}{n(\vec{x},\tau)} \int d^3\vec{p}\, \frac{p_i}{a m} \frac{p_j}{a m}\, f(\vec{x},\vec{p},\tau) - v^i(\vec{x},\tau)\, v^j(\vec{x},\tau)$

persion

tress tensor)

⋮

taking the moments of the Vlasov eq:

$$\frac{\partial}{\partial\tau}\delta(\vec{x},\tau) + \frac{\partial}{\partial x^i}\left[(1+\delta(x,\tau))\, v^i(\vec{x},\tau)\right] = 0 \quad \text{"continuity eq"}$$

$$\frac{\partial}{\partial\tau} v^i(\vec{x},\tau) + \mathcal{H} v^i(\vec{x},\tau) + v^k(\vec{x},\tau)\frac{\partial}{\partial x^k} v^i(\vec{x},\tau) +$$

$$+ \frac{1}{1+\delta}\frac{\partial}{\partial x^k}\left[(1+\delta)\,\sigma^{ik}(\vec{x},\tau)\right] = -\nabla^i \phi(\vec{x},\tau) \quad \text{"Euler eq."}$$

"pressure force" "gravitational force"

$$\frac{\partial}{\partial\tau}\sigma^{ik} + \dots = 0$$

$\dots = 0$ ← No source terms for moments above σ^{ij}

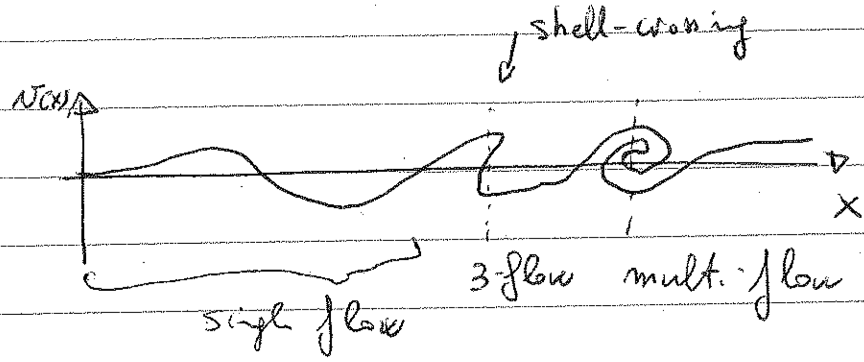
$$\nabla^2\phi = \frac{3}{2}\mathcal{H}^2\Omega_m\,\delta$$

The system continuity + Euler + Poisson in general is not closed: need a completion for σ^{ij}

$$\text{EG:}\quad \sigma^{ij} = \overset{\text{speed of sound}}{c_s^2}\,\delta_{ij}\,\delta \qquad \text{perfect fluid}$$

$\quad$ " + viscosity (bulk and shear) $\quad$ imperfect fluid

$\sigma^{ij} \simeq 0 \;\to\;$ single stream ~~approximation~~ flow



The first stage of the gravitational collapse, starting from linear initial conditions, takes place in the single stream regime $\sigma^{ij} \simeq 0$.

Later, multi-streaming appears, → relaxation → virialization. Difficult to treat (semi-)analytically

↳ N-body

$$\boxed{\text{Single stream approximation: } f(\vec{x},\vec{p},\tau) = \overbrace{\frac{1}{m}\rho(\vec{x},\tau)}^{n(\vec{x},\tau)} \times \delta_D\left(\vec{p} - a m \vec{v}(\vec{x},\tau)\right)}$$

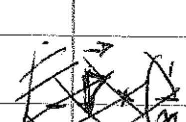
$\Rightarrow \sigma^{ij} = 0$ from the definition of the moments.

Later in these lectures we will discuss more on going beyond the SSA. But for the moment $\sigma^{ij}=0$.

<u>Vorticity:</u> $v^i(\vec{x},\tau) = v_\theta^i(\vec{x},\tau) + v_\omega^i(\vec{x},\tau)$

such that $\frac{\partial}{\partial x^i} v_\omega^i = 0$. Then $\omega^i \equiv \varepsilon^{ijk}\partial_j v^k = \varepsilon^{ijk}\partial_j v_\omega^k$

From Euler $\hookrightarrow$ $\frac{\partial}{\partial \tau}\omega^i + \mathcal{H}\omega^i - \varepsilon^{kji}\varepsilon^{lmk}\frac{\partial}{\partial x^j}(v_l\,\omega_m) = 0$

- In linear regime $\rightarrow$ $v_\omega^i \sim \frac{1}{a}$
- No source term (cannot be sourced by potential)
- sourced only after shell crossing by σ^{ij} 
  $\left[-\varepsilon^{ijk}\frac{\partial}{\partial x^j}\left(\frac{1}{n}\partial_l(n\sigma^{lk})\right)\right]$

$\hookrightarrow$ $\boxed{\text{set } v_\omega^i = 0 \text{ and describe } v^i \text{ by its divergence } \theta(\vec{x},\tau) \equiv \frac{\partial}{\partial x^i} v^i(\vec{x},\tau)}$

Fourier space $\quad \tilde{g}(\vec{k},\tau) \equiv \int d^3x \; g(\vec{x},\tau)\, e^{-i\vec{k}\cdot\vec{x}}$

$v^j(k) = -i \frac{k^j}{k^2}\,\theta(k) \qquad \left[\, g(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}\, \tilde{g}(\vec{k},\tau) \right]$

$\underbrace{\text{Continuity + Euler + Poisson}}$

$$\rightarrow \begin{cases} \dot{\delta}(\vec{k},\tau) + \theta(\vec{k},\tau) + I_{\vec{k};\vec{q}_1,\vec{q}_2}\, \alpha(q_1,q_2)\, \delta(q_1,\tau)\,\theta(q_2,\tau) = 0 \\ \dot{\theta}(\vec{k},\tau) + \mathcal{H}\,\theta(\vec{k},\tau) + \frac{3}{2}\mathcal{H}^2\Omega_m\,\delta(\vec{k},\tau) + \\ \qquad + I_{k;q_1,q_2}\, \beta(q_1,q_2)\,\theta(q_1,\tau)\,\theta(q_2,\tau) = 0 \end{cases}$$

where $\quad I_{k;q_1,\dots q_n} \equiv \int \frac{d^3q_1}{(2\pi)^3} \cdots \frac{d^3q_n}{(2\pi)^3}\, \delta_D\!\left(\vec{k} - \sum_i^n \vec{q}_i\right)$

$$\alpha(\vec{q}_1,\vec{q}_2) = \frac{(\vec{q}_1+\vec{q}_2)\cdot\vec{q}_2}{q_2^2}\,, \qquad \beta(\vec{q}_1,\vec{q}_2) = \frac{(q_1+q_2)^2\; \vec{q}_1\cdot\vec{q}_2}{2\, q_1^2 q_2^2} \quad \Bigg\}\ \text{Nonlinear coupling functions}$$

Linear theory : $\quad \alpha = \beta = 0 \quad \rightarrow \begin{cases} \dot{\delta} + \theta = 0 \\ \dot{\theta} + \mathcal{H}\theta + \frac{3}{2}\mathcal{H}^2\Omega_m\,\delta = 0 \end{cases}$

$$\rightarrow \boxed{\ddot{\delta} + \mathcal{H}\dot{\delta} = \frac{3}{2}\mathcal{H}^2\Omega_m\,\delta}$$

Eq. for matter perturbations
in the Newtonian limit $\quad k/\mathcal{H} \gg 1$

$$x = \ln a \,, \quad \delta(x) = \delta(x_{in})\, \underbrace{e^{\int_{x_{in}}^{x} f(x')\,dx'}}_{D(x,\, x_{in}) \text{ growth factor}} \quad (*)$$

(k- independent, in ΛCDM)

$$\frac{\partial}{\partial x} = \frac{1}{\mathcal{H}} \frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial x} \delta = f \delta = - \frac{\theta}{\mathcal{H}} \longrightarrow \boxed{\delta = - \frac{\theta}{f \mathcal{H}}}$$

$$\frac{\partial}{\partial x} f + f^2 + \left(1 + \frac{1}{\mathcal{H}} \frac{\partial \mathcal{H}}{\partial x}\right) f - \frac{3}{2} \Omega_m = 0$$

<u>EdS</u>: $\Omega_m = 1$, $\frac{1}{\mathcal{H}} \frac{\partial \mathcal{H}}{\partial x} = -\frac{1}{2}$ $\Rightarrow$ $f = $ 1 growing $= f_+$; $-\frac{3}{2}$ decaying $= f_-$

$\delta = - \frac{\theta}{\mathcal{H}}$ $\quad \delta > 0 \quad$ growing

$\delta = \frac{2}{3} \frac{\theta}{\mathcal{H}}$ $\quad \delta > 0 \quad$ decaying

COMPACT NOTATION

$$\eta = \ln \frac{D_+(z)}{D_+(z_{in})} = \int_{x_{in}}^{x} f(x')\,dx'$$

$$\varphi_a(\vec{k}, \eta) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = e^{-\eta} \begin{pmatrix} \delta(\vec{k}, \eta) \\ \dfrac{\theta(\vec{k}, \eta)}{f_+ \mathcal{H}} \end{pmatrix}$$

growing mode

- notice that $\varphi_a \propto u_a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the linear growing mode
- notice that $\varphi_a(\vec{k}, \eta) = \text{const}$ on the linear growing mode, because of the $e^{-\eta}$ factor.

$\rightarrow$ the continuity + Euler equations can be written in the compact form:

$$\boxed{\left(\delta_{ab}\partial_\eta + \Omega_{ab}\right)\varphi_b(\vec{k},\eta) = e^\eta I_{\vec{k},\vec{q}_1,\vec{q}_2}\, \gamma_{abc}(k,q_1,q_2)\varphi_b(q_1,\eta)\,\varphi_c(q_2,\eta)}$$

with $\Omega_{ab}(\eta) = \begin{pmatrix} 1 & -1 \\ -\frac{3}{2}\frac{\Omega_m}{f_+^2} & \frac{3}{2}\frac{\Omega_m}{f_+^2} \end{pmatrix}$

and the only non-vanishing elements of the "vertex" γ_{abc} are:

$$\gamma_{112}(k,q_1,q_2) = \frac{1}{2}\alpha(\vec{q}_1,\vec{q}_2) = \frac{1}{4}\frac{k^2+q_2^2-q_1^2}{q_2^2}$$

$$= \gamma_{121}(k,q_2,q_1) \qquad \vec{k}=\vec{q}_1+\vec{q}_2$$

$$\gamma_{222}(k,q_1,q_2) = \beta(\vec{q}_1,\vec{q}_2) = \frac{k^2(k^2-q_1^2-q_2^2)}{4q_1^2q_2^2}$$

Notice that we are using $\eta = \ln D_+^{(a)}$ as "time variable". If D_+ is also scale-dependent this is not possible any more. In that case one can use $\chi = \ln a$.

EXERCISE: show that in terms of χ and defining $\tilde{\varphi}_a = e^{-\chi}\begin{pmatrix} \delta \\ -\frac{\theta}{\mathcal{H}} \end{pmatrix}$ the eqs. take the form: $\rightarrow$

Note, no f!

$$(*)\quad \left(\delta_{ab}\,\partial_x + \tilde{\Omega}_{ab}\right)\tilde{\varphi}_b = e^{-x}\, I_{k,q_1 q_2}\, \gamma_{abc}\, \tilde{\varphi}_b \tilde{\varphi}_c$$

with $$\tilde{\Omega}_{ab} = \begin{pmatrix} 1 & -1 \\ -\frac{3}{2}\Omega_m & 2 + \frac{d\log H}{dx} \end{pmatrix}$$

It can be used for instance for CDM + neutrinos.

$$\nabla^2\phi = \frac{3}{2}H^2\left(\Omega_c\delta_c + \Omega_\nu\delta_\nu\right) = \frac{3}{2}H^2\underbrace{\Omega_c\left(1 + \frac{\Omega_\nu\delta_\nu}{\Omega_c\delta_c}\right)}_{\equiv\,\bar{\Omega}_c(\vec{k},\eta)}\delta_c$$

take $\frac{\Omega_\nu\delta_\nu}{\Omega_c\delta_c}$ from linear th., then the eq for δ_c, θ_c

follow $(*)$ with $\Omega_m = \bar{\Omega}_c(\vec{k},\eta)$.

Perturbative solution of the eqs. of motion

$$(\delta_{ab}\partial_\eta + \Omega_{ab})\varphi_b(k,\eta) = I_{k q_1 q_2}\, e^{\eta}\, \gamma_{bcd}\, \varphi_c(q_1,\eta)\varphi_d(q_2,\eta)$$

|Linear order| $\gamma = 0$ $\quad (\delta_{ab}\partial_\eta + \Omega_{ab})\varphi_b(k,\eta) = 0$

Solution: $\boxed{\varphi_a^{(1)}(k,\eta) = g_{ab}(\eta-\eta_{in})\,\varphi_b^{in}(k)}$ ← initial condition (see later)

where $g_{ab}(\eta)$ is the linear propagator and satisfies

$$\boxed{(\delta_{ab}\partial_\eta + \Omega_{ab})\,g_{bc}(\eta) = \delta_{ac}\,\delta(\eta)}$$

How to compute the linear propagator (alternative: Laplace transf.)
FOR CONSTANT Ω_{ab}

1) find the eigenvalues and eigenvectors of Ω_{ab}

$$\Omega_{ab}\, u_b^A = \lambda^A u_a^A \qquad A = 1,2$$

2) ~~complete~~ Find the matrices

$$M^A \equiv \lim_{\lambda\to-\lambda^A} (\lambda+\lambda^A)\left[\lambda\cdot\mathbb{1} + \Omega\right]^{-1} \qquad A=1,2$$

3) the propagator is given by: $g_{ab}(\eta) = \left(M_{ab}^1 e^{-\lambda^1\eta} + M_{ab}^2 e^{-\lambda^2 \eta}\right) \times \theta(\eta)$

This procedure can be generalized to multi-species

n species $\to$ $2n$ equations of motion $\to$ $2n \times 2n$ matrices...

In the EdS ($\Omega_m = 1$) case, for a single species, the propagator is:

$$g_{ab}(\eta) = \theta(\eta)\left[\frac{1}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix} e^{-\frac{5}{2}\eta}\right]$$

growing mode $u_a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

decaying mode $v_a = \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$

Notice that
$$\begin{cases} g_{ab}(\eta \to 0^+) = \mathbb{1}_{ab} \\ g_{ab}(\eta - \eta')\, g_{bc}(\eta' - \eta'') = g_{ac}(\eta - \eta'') \end{cases}$$

<u>Initial condition in the adiabatic growing mode</u>: $\varphi_a^{in}(k) = \varphi^{in} u_a$, $u_a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\eta_{in} \to -\infty \text{ formally, in practice, } z_{in} \simeq 50-100}$ from CAMB, ...

At later times, in linear approx: $\varphi_a^{(1)}(k,\eta) = g_{ab}(\eta)\varphi_b^{in} =$

$$= \varphi^{in}(k)\, u_a$$

$$\left[\frac{1}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \frac{1}{5}\begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right]$$

Remember: $\varphi_a(k;\eta) \equiv e^{-\eta}\begin{pmatrix} \delta \\ -\frac{\vartheta}{\mathcal{H}f_+} \end{pmatrix}$ $\quad e^{-\eta} = \frac{D^+_{in}}{D^+(\eta)}$

$\rightarrow$ constant in linear ~~theo~~ approx. Ok!

- In cosmologies $\neq$ EdS, with a time dep growth function

e.g. ΛCDM $\Omega_m = \Omega_m(\eta) \neq 1$ we can still give an analytic expression for the propagator

[see JCAP 10(2006)036, Appendix A]

~~However~~ In these cosmologies, the Ω matrix is time dep:

$$\begin{pmatrix} 1 & -1 \\ -\frac{3}{2}\frac{\Omega_m}{f_+^2} & \frac{3}{2}\frac{\Omega_m}{f_+^2} \end{pmatrix}$$

Notice that, as for EdS, $\Omega \cdot u = 0$, that is, the linear evolution of the growing mode is the same as for EdS, since $\varphi_a(k;\eta) = e^{-\eta}\begin{pmatrix} \delta \\ -\frac{\vartheta}{\mathcal{H}f_+} \end{pmatrix}$ and $\delta = -\frac{\vartheta}{\mathcal{H}f_+}$ ~~as~~ on the linear growing mode.

So, it is only the evolution of the decaying mode which is modified.

Therefore, a commonly adopted approximation is to set $\frac{\Omega_m}{f^2} \simeq 1$ in the Ω matrix and use the same propagator as for ~~the~~ EdS.

This gives modification only at loop orders, which are however at the subpercent level up to $k \lesssim 0.4\, h/\mathrm{Mpc}$ (see JCAP 10 (2006) 036)

- In cosmologies with scale and time-dependent growth functions, use the $x = \log a$ variable. The propagator can be computed but its use is impractical. Better other approaches, as the TRG.

The perturbative expansion of the solution of the eqs. of motion is an expansion in powers of the initial condition field: $\varphi^{in}(k)$.

We will always consider initial growing modes: u^a

$$\varphi_a(\vec{k};\eta) = \sum_1^\infty{}_n \varphi_a^{(n)}(\vec{k};\eta) = \sum_1^\infty{}_n \, I_{k,q_1\cdots q_n} F_a^{(n)}(\vec{k};\vec{q}_1,\ldots\vec{q}_n,\eta)$$
$$\varphi^{in}(\vec{q}_1)\cdots\varphi^{in}(\vec{q}_n)$$

The linear solution $\varphi_a^{(1)}(\vec{k};\eta) = u_a \varphi^{in}(\vec{k})$ implies

$$F_a^{(1)}(\vec{k};\cancel{\eta}) = u_a$$

inserting it at the RHS of the eq. of motion and using eqn. the propagator, we have.

$$\varphi_a^{(2)}(\vec{k},\eta) = \int_{\eta_{in}}^{\eta} ds \, g_{ab}(\eta-s) e^{s} I_{k,q_1,q_2} \gamma_{bcd}(\vec{k},\vec{q}_1,\vec{q}_2) u_c u_d$$
$$\varphi^{in}(\vec{q}_1)\varphi^{in}(\vec{q}_2)$$

$$\hookrightarrow F_a^{(2)}(\vec{k};\vec{q}_1,\vec{q}_2;\eta) = \int_{\eta_{in}}^{\eta} ds \, g_{ab}(\eta-s) e^{s} \gamma_{bcd}(k;\vec{q}_1,\vec{q}_2) u_c u_d$$

At third order we get:

$$\varphi^{(3)}_a(\vec{k};\eta) = \int_{\eta_{in}}^{\eta} ds\, e^{s} g_{ab}(\eta-s)\, I_{k,q_1,q_2}\, \gamma_{bcd}(\vec{k};\vec{q}_1,\vec{q}_2) \times$$

$$\times \left(\varphi^{(1)}_c(\vec{q}_1,s)\varphi^{(2)}_d(\vec{q}_2,s) + \varphi^{(2)}_c(\vec{q}_1,s)\varphi^{(1)}_d(\vec{q}_2,s)\right)$$

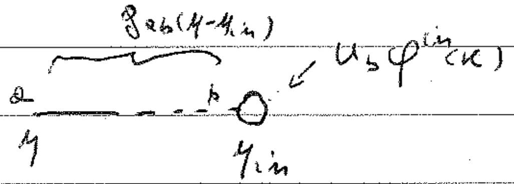
$$= 2\int_{\eta_{in}}^{\eta} ds\, e^{s} g_{ab}(\eta-s)\, I_{kq_1,q_2}\, \gamma_{bcd}(\vec{k};\vec{q}_1,\vec{q}_2)$$

$$u_c\varphi^{in}(\vec{q}_1)\int_{\eta_{in}}^{s} ds'\, g_{de}(s-s')\, e^{s'} \underset{I_{q_2,p_1,p_2}}{\wedge} \gamma_{efg}(\vec{q}_2;\vec{p}_1,\vec{p}_2)\, u_f u_g\, \varphi^{in}(\vec{p}_1)\varphi^{in}(\vec{p}_2)$$

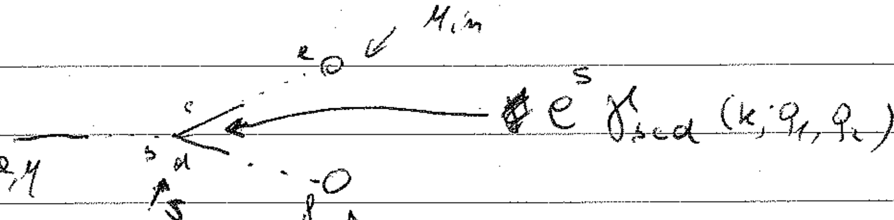
$$\to F^{(3)}_a(\vec{k};\vec{q}_1,\vec{q}_2,\vec{q}_3) = \cdot\cdot\,(\text{symmetrize over } q_1, q_2, q_3 \ldots)$$

Notice the tree structure, that can be represented diagrammatically

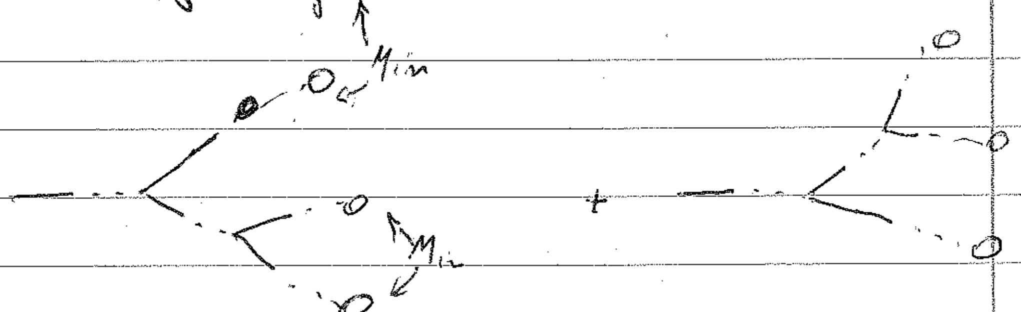
$\varphi^{(1)}_a(\vec{k};\eta)$



$\varphi^{(2)}_a(\vec{k};\eta)$



$\varphi^{(3)}_a(\vec{k};\eta)$



$\varphi^{(4)}_a(\vec{k};\eta) = \ldots$

EXERCISE: Compute $F_a^{(2)}(\vec{k};\vec{q}_1,\vec{q}_2)$ in the $\eta_{in}\to-\infty$ limit ($z_{in}\to\infty$)

$$\to\quad F_1^{(2)}(\vec{k};\vec{q}_1,\vec{q}_2) = \frac{1}{2}\left(\frac{k\cdot q_1}{q_1^2}+\frac{k\cdot q_2}{q_2^2}\right)\frac{5}{7}+\frac{1}{2}\frac{k^2\,\vec{q}_1\cdot\vec{q}_2}{q_1^2q_2^2}\frac{2}{7}$$

(a.k.a. F_2)

$$F_2^{(2)}(\quad) = \frac{1}{2}(\quad " \quad)\frac{3}{7}+\frac{1}{2}\frac{k^2\,\vec{q}_1\cdot\vec{q}_2}{q_1^2q_2^2}\frac{4}{7}$$

(a.k.a. G_2)

The power spectrum

modulus (isotropy)

$$\langle\varphi_a(\vec{k},\eta)\,\varphi_b(\vec{k}',\eta')\rangle = (2\pi)^3\,\delta_D(k+k')\,P_{ab}(k;\eta,\eta')$$

if we take equal times (usual choice) we indicate

as $P_{ab}(k;\eta,\eta') = P_{ab}(k;\eta)$

Given the definition of φ_a, the usual ~~PS~~ density

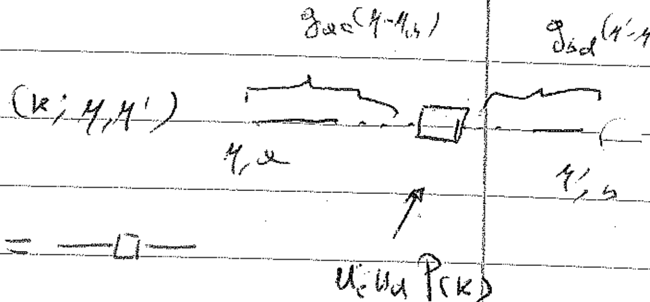
PS is $P_{\delta\delta}(k;\eta) = e^{2\eta}\,P_{11}(k;\eta) = D_+^2(z)\,P_{11}(k;\eta)$

Linear PS: $\langle\varphi_a^{(1)}(k;\eta)\,\varphi_b^{(1)}(k;\eta)\rangle = u_a u_b\langle\varphi^{in}(k)\varphi^{in}(k')\rangle$

$$= (2\pi)^3\,\delta(\vec{k}+\vec{k}')\,u_a u_b\,P(k)$$

initial (linear) PS

diagrammatic representation: $P_{ab}(k;\eta,\eta')$



If the primordial statistics is gaussian, then it is all we need to compute correlators at any order.
~~Otherw~~ Otherwise, we also need higher order correlators at η_{in}.

Primordial bispectrum $\langle \varphi_a^{in}(\vec k)\,\varphi_b^{in}(\vec k')\,\varphi_c^{in}(\vec k'')\rangle =$

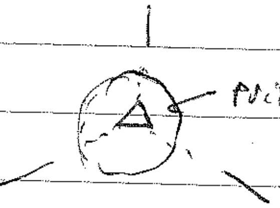
$$= (2\pi)^3\,\delta_D(\vec k+\vec k'+\vec k'')\,u_a u_b u_c\, B(k,k',k'')$$

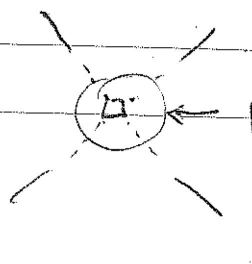
Primordial Trispectrum $\langle \varphi_a^{in}(k)\,\varphi_b^{in}(k')\,\varphi_c^{in}(k'')\,\varphi_d^{in}(k''')\rangle -$

$$\langle\varphi_a\varphi_b\rangle\langle\varphi_c\varphi_d\rangle - \langle\varphi_a\varphi_c\rangle\langle\varphi_b\varphi_d\rangle$$

$$-\langle\varphi_a\varphi_d\rangle\langle\varphi_b\varphi_c\rangle = (2\pi)^3\,\delta_D(k+k'+k''+k''')$$

$$T(k,k',k'',k''')\,u_a u_b u_c u_d$$

B_{abc} →  ← primordial bispectrum

T_{abcd} →  ← primordial trispectrum

In the following, we will assume that the primordial statistics is gaussian. [For the non-gaussian case we have extra diagrams, see 0912.4276]

Then all correlators of an even number of primordial fields can be split in products of primordial PS's.

All odd correlators vanish.

EXERCISE: compute the Bispectrum at the lowest order for gaussian initial conditions.

The PS at the first non linear order

(1-loop, 3rd order PT, ...)

$$\langle \varphi_a(\vec{k},\eta)\,\varphi_b(\vec{k}',\eta')\rangle = \underbrace{\langle \varphi_a^{(1)}\varphi_b^{(1)}\rangle}_{P^{lin}} + \underbrace{\langle \varphi_a^{(1)}\varphi_b^{(3)}\rangle}_{\text{"}P_{13}\text{"}/2} +$$

$$+ \underbrace{\langle \varphi_a^{(3)}\varphi_b^{(1)}\rangle}_{\text{"}P_{13}\text{"}/2} + \underbrace{\langle \varphi_a^{(2)}\varphi_b^{(2)}\rangle}_{\text{"}P_{22}\text{"}}$$

Take $\eta=\eta'$ for simplicity

$$=(2\pi)^3\,\delta_D(k+k')\,P_{ab}^{1\text{-loop}}(k,\eta)$$

$$=(2\pi)^3\,\delta_D(k+k')\Big\{ u_a u_b P(k) + \Big[u_a\, I_{k,q_1q_2q_3}\, F_b^{(3)}(k;q_1,q_2,q_3;\eta)$$

$$\langle \varphi(\vec{k})\varphi(\vec{q}_1)\varphi(\vec{q}_2)\varphi(\vec{q}_3)\rangle + (a\leftrightarrow b)\Big] +$$

$$+ I_{k,q_1,q_2} I_{-k,p_1,p_2} F_a^{(2)}(k,q_1,q_2) F_b^{(2)}(-k,p_1,p_2) \langle \varphi(q_1)\varphi(q_2)\varphi(p_1)\varphi(p_2)\rangle \Big]$$

$$+ \mathcal{O}\langle\varphi^6\rangle$$

$$= (2\pi)^3 \delta_D(k+k') \Big\{ u_a u_b P(k) + \Big[4 \int_{\eta_{in}}^{\eta} ds \int_{\eta_{in}}^{s} ds' \, e^{s+s'}$$

using ~~next~~ propagator, vertices, PS, $B_{in} = T_{in} = 0$,

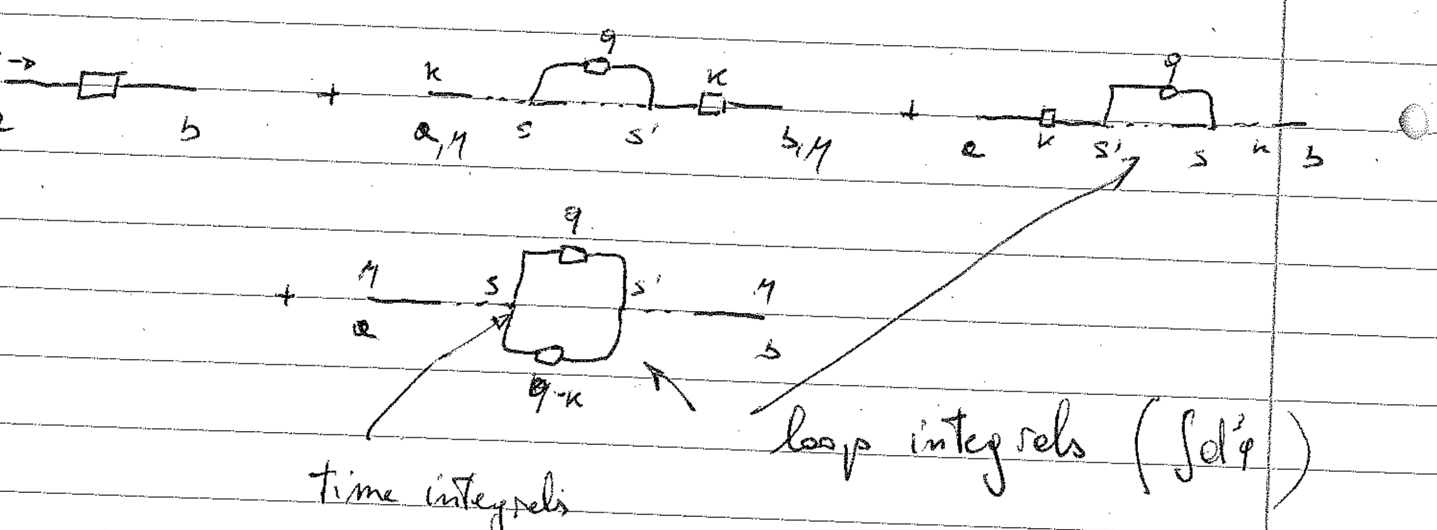
$$\int \frac{d^3q}{(2\pi)^3} g_{ac}(\eta - s) \gamma_{cde}(k, q, |k-q|) \, g_{ef}(s-s') \gamma_{fgh}(|k-q|, q, k) u_d u_g u_h$$

$$P(k) P(q) u_b \quad + \quad (a \leftrightarrow b) \Big] +$$

$$+ 2 \int_{\eta_{in}}^{\eta} ds \int_{\eta_{in}}^{\eta} ds' \, e^{s+s'} \int \frac{d^3q}{(2\pi)^3} \, g_{ac}(\eta - s) \, g_{bd}(\eta - s') \gamma_{cef}(k, q, |k-q|) \gamma_{dgh}(k, q, |k-q|)$$

$$u_e u_f u_g u_h \, P(q) \, P(|k-q|) \Big\}$$

Diagrammatic representation:



EXERCISE: Compute $P_{11}^{1\text{-loop}}(k,\eta)$ and check that is given by

$\eta_{in} \to -\infty$

$$P_{11}^{1\text{-loop}}(k;\eta) = \overset{P(z_{in})}{P(k)} + \left(P^{(13)}(k) + P^{(22)}(k)\right) \underbrace{e^{2\eta}}_{\frac{D_+^2(z)}{D_+^2(z_{in})}}$$

with
$$P^{(13)}(k) = P(k)\,\frac{k^3}{252}\,\frac{1}{(2\pi)^2}\,\times$$
$$\times \int_0^\infty dr\, P(kr)\left[\frac{12}{r^2} - 158 + 100r^2 - 42r^2 + \frac{3}{r^3}(r^2-1)^3(2+7r^2)\ln\left|\frac{r+1}{r-1}\right|\right]$$

$$P^{(22)}(k) = \frac{k^3}{98}\,\frac{1}{(2\pi)^2}\int_0^\infty dr\, P(kr)\int_{-1}^{1} dx\, P\left(k\sqrt{1+r^2-2rx}\right)\times$$
$$\times\frac{(3r+7x-10rx^2)^2}{(1+r^2-2rx)^2}$$

———— o ———— o ————

To get the PS for δ, the expression above has to be multiplied by $\left(\frac{D_+(z_{out})}{D_+(z_{in})}\right)^2$.

IR and UV behavior of the loop integrals.

Depending on the k-dependence of the linear PS, the loop integrals can exhibit divergences in the IR & in the UV. It is very instructive to understand their physical origin.

Assume $P(k) \sim k^{n}$ for $k \to 0$ IR

$\sim k^{m}$ for $k \to \infty$ UV

$\boxed{\text{IR:}}$ The $P^{(13)}$ term, when the loop momenta $q \to 0$

gives $P^{(13)}_{q \to 0}(k,\eta) \sim e^{2\eta} P(k) \int \frac{d^3q}{(2\pi)^3} P(q) \left(-\frac{1}{3}\frac{k^2}{q^2} + O(1) \right)$

The $P^{(22)}$ term is more subtle, as there are two IR contribution $\int d^3q \; P(q) P(|k-q|) \; G(\vec{k},\vec{q})$

$\vec{q} \to 0$, $|q-k| \to k$; $|\vec{q}-\vec{k}| \to 0$, $\vec{q} \to \vec{k}$

Summing carefully the two contribution, we get

$P^{(22)}(k,\eta) \sim e^{2\eta} P(k) \int \frac{d^3q}{(2\pi)^3} P(q) \left(+\frac{1}{3}\frac{k^2}{q^2} + O(1) \right)$

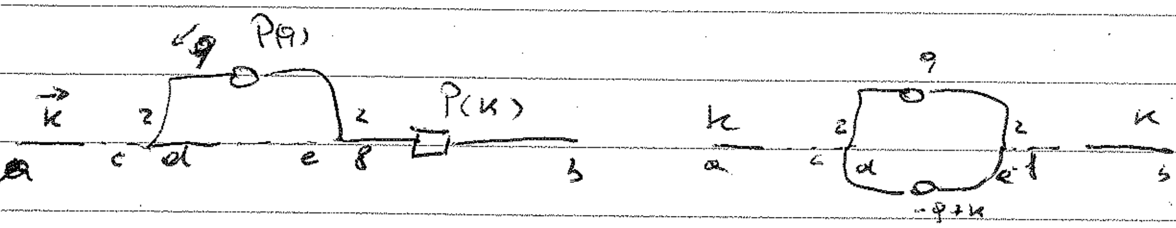
→ ~~exact~~ the leading IR divergence exactly cancels in the sum, leaving a global IR behaviour $\boxed{\int d^3q\, P(q)\ ,\ \text{safe if } n > -3}$

$(P_{\Lambda CDM} \sim q^{n_s} \quad n_s = 0.9603)$

The cancellation of the $\frac{k^2}{q^2}$ term is not accidental:

it is a consequence of "Galilean" invariance.

the $\frac{k^2}{q^2}$ term comes from long wavelength velocity perturbation



$$\gamma_{cde}(k, -q+k, -q) \xrightarrow[q \ll k]{} -\frac{1}{2}\frac{\vec{k}\cdot\vec{q}}{q^2}\delta_{cd} \qquad \left(v^j(q) = -i\frac{q^j \theta_q}{q^2}\right.$$

↑ velocity

but such a coherent motion cannot have consequences on an equal time short distance ($\sim \frac{1}{k}$) correlator $\longrightarrow$ $\frac{k^2}{q^2}$ term must cancel.

The remaining term $\sim \int d^3q\, P(q)$ is the variance of the density field. A long wavelength density field has an effect on short distance correlators,

$(\delta \sim \nabla^2\phi)$

$\partial_\tau \delta + \vec{\nabla}((1+\delta)\vec{v}) = 0$

$\delta_\ell = \delta_L + \delta_S \rightarrow \partial_\tau \delta_S + (1+\delta_L)\vec{\nabla} v_L \simeq 0$

if its variance diverges, then the correlator diverges too.

UV when loop momenta $q \gg k$ (external momentum)

$$J^1 \sim \frac{k \cdot q}{q^2} \rightsquigarrow \Delta P^{1\text{-loop}} \sim \int d^3q \, P(q) \frac{k^2}{q^2} \quad (q \gg k)$$

$$\boxed{\text{UV converges if } n < -1} \quad P_{\Lambda CDM} \sim q^{n_s - 4} \log q$$

what happens if $n \geq -1$? $\langle v^2 \rangle = \int d^3q \, v^i(q) v^i(-q)$

$$= + \int d^3q \, \frac{\theta(q)\theta(-q)}{q^2} = \int d^3q \, \frac{P(q)}{q^2}$$

↳ the small scale velocity dispersion diverges

↳ break down of the single stream approximation

$(5 \sim \langle v^2 \rangle \neq 0)$

At higher loops the UV convergence worsens

$$n < -3 + \frac{2}{p}$$

p ← loop order

UV regularization needed!

Possible solutions:

- resummation (partial resummation)
- hybrid approaches (take the UV from N-body)
- different PT schemes

Performance of Standard PT:

see for instance FIGS 2,3 of 1309.3308

- doubts on the convergence at $z \leq 1-2$
  → asymptotic series?

% level up to $k \sim 0.3\ h/Mpc$

- 1 and 2-loop OK above $z = 2,\ 0.8$, respectively

(linear theory @ $z=0$ ok only up to $k \sim 0.02\ h/Mpc$)

- Need to go beyond PT for lower redshifts and for smaller scales

A non-perturbative scheme to solve the eqs of motion

Time flow equations → TRG

- Particularly suited for cosmologies with a scale-dependent growth factor. (massive neutrinos, coupled quintessence, ...)

- No need to compute the linear propagator
  → no approximation on the decay mode

- non-perturbative ($O(g^n)$ terms summed at all orders)
however, going beyond the lowest level of approximation is difficult / numerically expensive

Use $\chi = \ln a$ $(a_0 = 1)$ as the "time" variable.

$$\hookrightarrow \partial_\chi \varphi_a(k;\chi) = -\underbrace{\Omega_{ab}(k,\chi)}_{\text{space and time-dep } \Omega\text{'s}}\varphi_b(k;\chi) + e^{\chi}\overbrace{\int}^{I_{k,q,p}} \gamma_{abc}(k,q,p)\,\varphi_b(q,\chi)\,\varphi_c(p,\chi)$$

$$\hookrightarrow \Omega_{ab}(k;\chi) = \begin{pmatrix} 1 & -1 \\ -\frac{3}{2}\Omega_m(\chi)(1+B(k,\chi)) & 2+\frac{\partial \ln H}{\partial \chi}+A(k;\chi) \end{pmatrix} \begin{array}{l} \leftarrow \text{continuity eq.} \\ \leftarrow \text{Euler eq.} \end{array}$$

$A(k,x)$ and $B(k;x)$ are function parameterizing the departure from a pure ΛCDM cosmology through the Euler equation (the continuity eq → particle number conservation, is untouched)

$$\frac{\partial}{\partial \tau}\theta + \mathcal{H}\,(1+A(k;\tau))\,\theta(k;\tau) + \frac{3}{2}\Omega_m(\tau)\mathcal{H}^2(1+B(k;\tau))\delta$$

$$+ \int I_{k,q,p}\, \beta(\vec{q},\vec{p})\,\theta(\vec{q},\tau)\,\theta(\vec{p},\tau) = 0$$

A : modification of the geodesic equation

(ex: Scalar-Tensor theories in the Einstein frame)

B : modification of the Poisson equation

(ex: massive neutrinos, quintessence, ...)

$$\frac{3\mathcal{H}^2}{2}\sum_i \Omega_i\,\delta_i = \frac{3}{2}\mathcal{H}^2\,\Omega_c\,\delta_c\,\underbrace{\left(1+\sum_{i\neq c}\frac{\Omega_i\,\delta_i}{\Omega_c\,\delta_c}\right)}_{1+B(k,\tau)}$$

cold DM

Notice that the non-linear vertices are untouched.

Screening ~~mod~~ models (chameleon, symmetron, ...)

would also require more non-linear terms

$\hookrightarrow$ not considered here

The idea is simply to take χ-derivatives of the correlators: $\langle \varphi_a(k;\chi)\varphi_b(-k;\chi)\rangle' = (2\pi)^3 P_{ab}(k;\chi)$

(where $\langle\ \rangle' = \dfrac{\langle\ \rangle}{\delta_D(\underset{=0}{\text{overall momentum}})}$)

symbolically

$\hookrightarrow \partial_\chi \langle \varphi_a \varphi_b\rangle' = \langle(\partial_\chi \varphi_a)\varphi_b\rangle' + \langle \varphi_a \partial_\chi \varphi_b\rangle' =$

$= -\Omega_{ac}\langle \varphi_c\varphi_b\rangle' - \Omega_{bc}\langle\varphi_a\varphi_c\rangle' +$

$+ e^\chi \gamma_{ade}\langle\varphi_d\varphi_e\varphi_b\rangle' + e^\chi \gamma_{bde}\langle\varphi_a\varphi_d\varphi_e\rangle'$

where as before, repeated indices are summed over, and

$\Omega_{ac}\langle\varphi_c\varphi_b\rangle' = \Omega_{ac}(k;\chi)\,\langle\varphi_c(\vec{k},\chi)\varphi_b(-\vec{k},\chi)\rangle'$,

$\gamma_{ade}\langle\varphi_d\varphi_e\varphi_b\rangle' = I_{k,q,p}\ \gamma_{ade}(k,q,p)\ \times$

$\times \langle\varphi_d(q,\chi)\varphi_e(p,\chi)\varphi_b(-k,\chi)\rangle'$

and so on

the eq. is not closed. we need an eq for $\langle\varphi\varphi\varphi\rangle$

$$\partial_\chi \langle \varphi_a(q_1,\chi)\varphi_b(q_2,\chi)\varphi_c(q_3,\chi)\rangle' =$$

$$- \Omega_{ad} \langle \varphi_d \varphi_b \varphi_c \rangle' - \Omega_{bd} \langle \varphi_a \varphi_d \varphi_c \rangle' -$$

$$- \Omega_{cd} \langle \varphi_a \varphi_b \varphi_d \rangle' +$$

$$+ e^\chi \gamma_{ade} \langle \varphi_d \varphi_e \varphi_b \varphi_c \rangle' + e^\chi \gamma_{bde} \langle \varphi_a \varphi_d \varphi_e \varphi_c \rangle'$$

$$+ e^\chi \gamma_{cde} \langle \varphi_a \varphi_b \varphi_d \varphi_e \rangle'$$

The system is not closed yet, we need the time evolution of $\langle\varphi\varphi\varphi\varphi\rangle'$.

First, we isolate the connected part of the 4-point function, the TRISPECTRUM

TRISPECTRUM

$$\langle \varphi_a(k,\chi)\varphi_b(q,\chi)\varphi_c(p,\chi)\varphi_d(r,\chi)\rangle = \delta_D(k+p+q+r)(2\pi)^3 T_{abcd}(k,q,p,r;\chi)$$

$$+ (2\pi)^6 \Big[\delta_D(k+q)\delta_D(p+r)\, P_{ab}(k,\chi) P_{cd}(p,\chi) + \delta_D(k+p)\delta_D(q+r)\, P_{ac}(k,\chi) P_{bd}(q,\chi)$$

$$+ \delta_D(k+r)\delta_D(q+p)\, P_{ad}(k,\chi) P_{bc}(q,\chi) \Big]$$

Truncation: set $\Phi_{abcd} = 0$

↳ closed system of equations

$$
\begin{cases}
\partial_x P_{ab}(k;x) = -\Omega_{ac}(k;x)\, P_{cb}(k,x) - \Omega_{bc}(k;x)\, P_{ac}(k,x) \\
\quad + e^{x} \underbrace{I_{k,q,p}}_{\text{momentum integral}} \Big\{ \gamma_{acd}(k,q,p)\, B_{bcd}(k,q,p;x) + \\
\quad + B_{acd}(k,q,p;x)\, \gamma_{bcd}(k,q,p) \Big\} \\[1em]
\partial_x B_{abc}(k,q,p;x) = -\Omega_{ad}(k;x)\, B_{dbc}(k,q,p;x) \\
\quad - \Omega_{bd}(q;x)\, B_{adc}(k,q,p;x) - \Omega_{cd}(p;x)\, B_{abd}(k,q,p;x) \\
\quad + 2 e^{x} \Big\{ \gamma_{ade}(k,q,p)\, P_{db}(q;x)\, P_{ec}(p,x) + \\
\quad + \gamma_{bde}(q,p,k)\, P_{dc}(p,x)\, P_{ea}(k,x) + \\
\quad + \gamma_{cde}(p,k,q)\, P_{da}(k,x)\, P_{eb}(q,x) \Big\}
\end{cases}
$$

momentum integral

Notice that the PS and the bispectrum at the rhs are time-dep, nonlinear ones.

To ~~see~~ understand to what approximation the solution of the system above corresponds to, we notice that it has the FORMAL SOLUTION:

$$P_{ab}(k;x) = g_{ac}(k;x,x_{in})\, g_{bd}(k;x,x_{in})\, P_{cd}(k;x_{in})$$

$$+ \int_{x_{in}}^{x} ds\, e^{s}\, I_{k,q,p} \left\{ g_{ae}(k;x,s)\, g_{bf}(k;x,s) \left[\gamma_{ecd}(k,q,p)\, B_{fcd}(k,q,p;s) + (e \leftrightarrow f) \right] \right\}$$

$$B_{abc}(k,q,p;x) = g_{ad}(k;x,x_{in})\, g_{be}(q;x,x_{in})\, g_{cf}(p;x,x_{in})\, B_{def}(k,q,p;x_{in}) +$$

$$2 \int_{x_{in}}^{x} ds\, e^{s}\, g_{ad}(k;x,s)\, g_{be}(q;x,s)\, g_{cf}(p;x,s) \times$$

$$\times \left[\gamma_{dgh}(k,q,p)\, P_{ge}(q;s)\, P_{hf}(p;s) + \right.$$

$$+ \gamma_{egh}(q,p,k)\, P_{gf}(p;s)\, P_{hd}(k;s) +$$

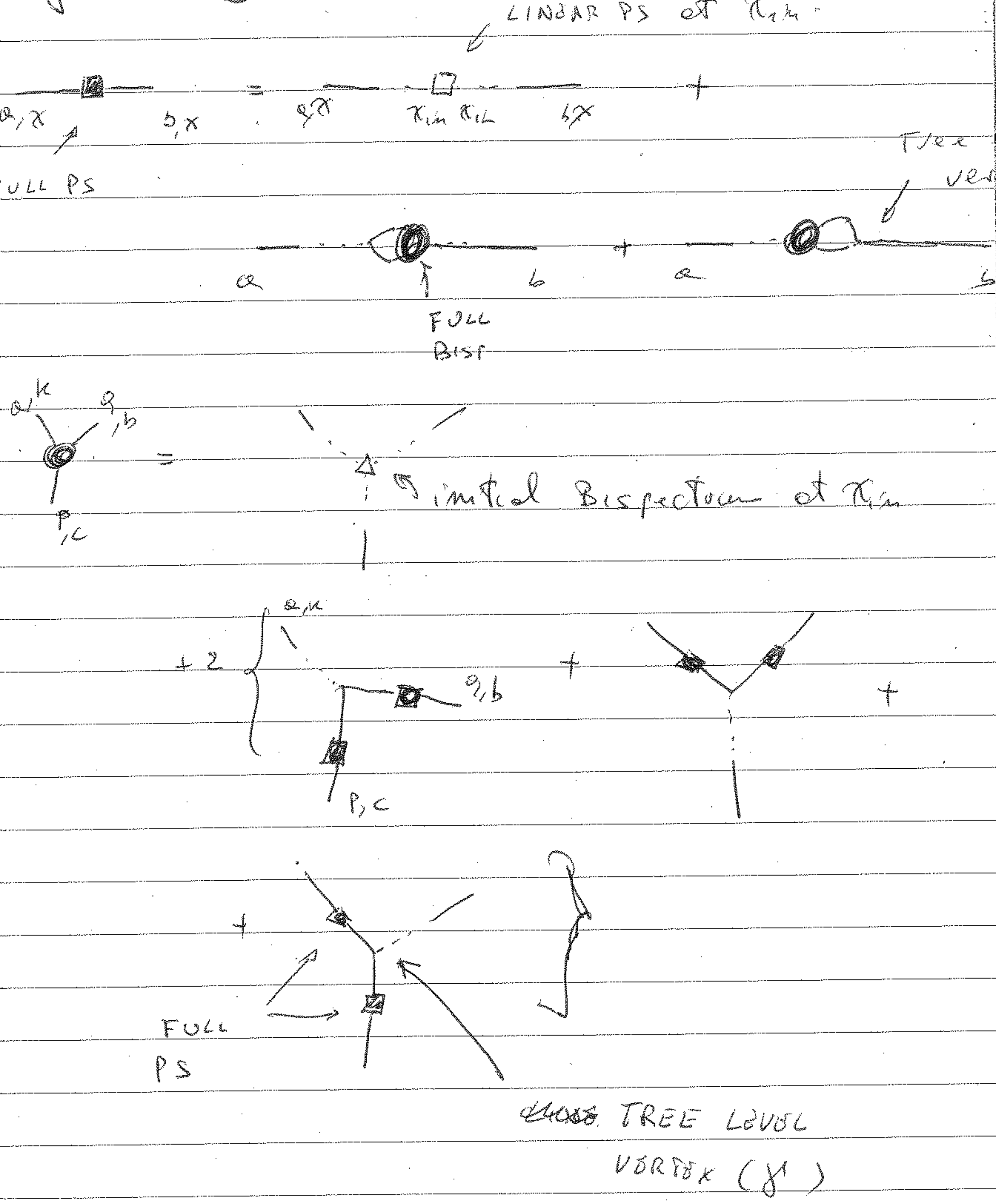
$$\left. + \gamma_{fgh}(p,k,q)\, P_{gd}(k,s)\, P_{he}(q,s) \right]$$

it can be checked by using the property of the linear propagator:

$$\partial_x g_{ab}(k;x,x') = -\Omega_{ac}(k,\eta)\, g_{cb}(k;x,x')$$

$$\lim_{x' \to x^-} g_{ab}(k;x,x') = \delta_{ab}$$

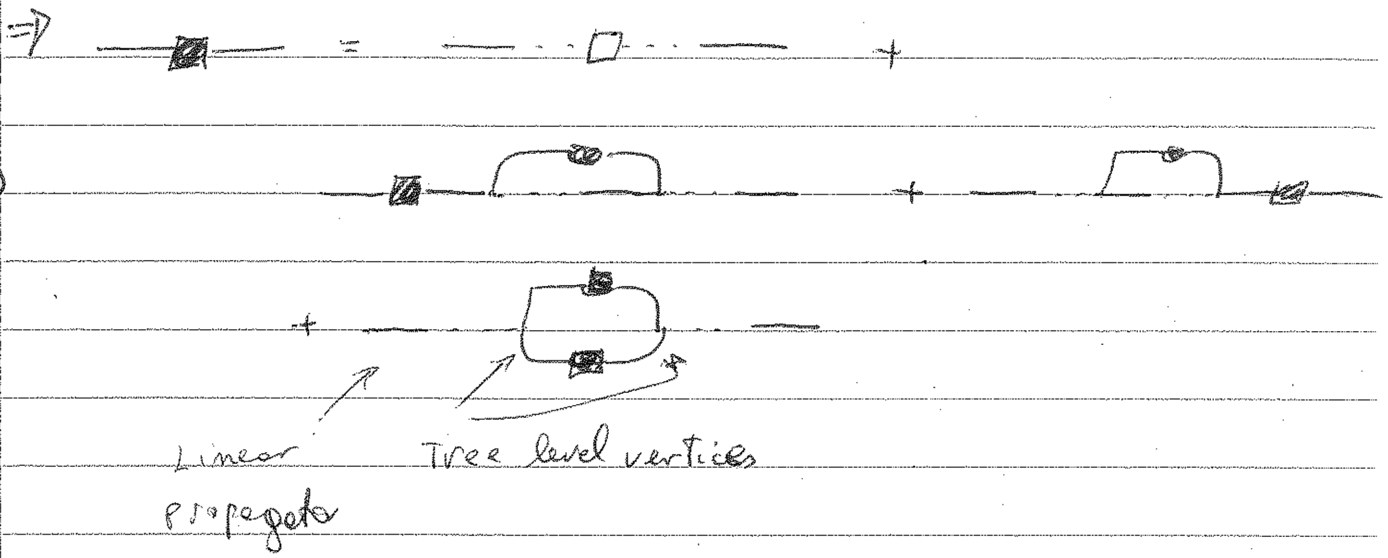
Diagrammatically:



Doing one step further, and setting to zero the initial bispectrum (gaussian initial conditions)

$\Rightarrow$

(*)

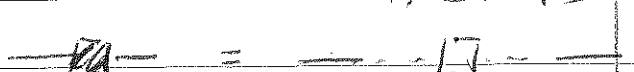


This diagrammatic expression is fully equivalent to the TRG equation with $B_{esc} = 0$ $\forall x$ and $B_{esc} = 0$ for $x = x_{in}$

- Solving it in the differential form is better because we don't need the explicit form for the propagator $\rightarrow$ NO NEED TO APPROXIMATE $\Sigma_{ab}(...)$
- Looking at the diagrammatic expression (*) helps understanding the relation to standard PT.

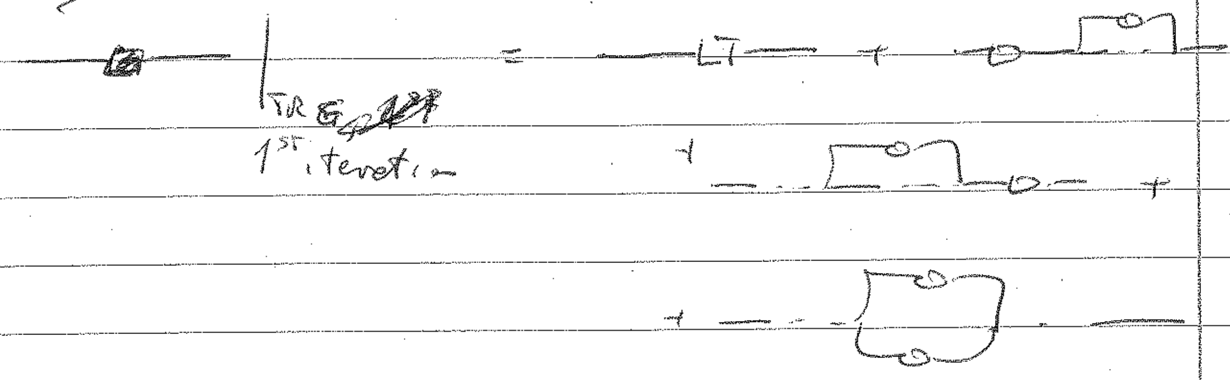
$\longrightarrow$

Linear PS

Iterative solution: set 

at the RHS, send $\chi_{in} \to \infty$

↳

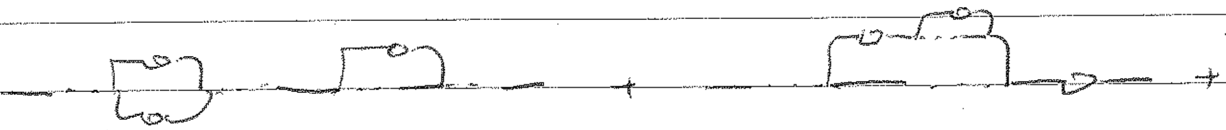


~~TREE~~ 1st iteration

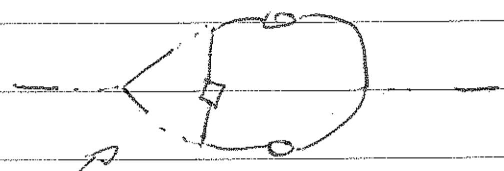
$= PS^{1\text{-loop}}$

So, setting the running PS to the linear one is equivalent to the 1-loop result.

Second iteration: get new contributions such as



get <u>some</u> 2-loop contributions, but not all; e.g. not



vertex correction

Standard PT and TRG truncated at $\Phi=0$

start to differ at 2-loop order.

Higher orders in the TRG iteration ~~g~~

contain all orders in standard PT (but no vertex

renormalization!). All these terms are automatically

~~included~~ computed by solving the ~~eq~~ TRG equat

in differential form.

In summary, TRG with $\Phi=0$ useful for

cosmologies with $\Omega_{ab}(k,\eta)$, no need to approximate

the η dependence. If the ~~η~~ k-dependence is

to be taken into account exactly, then the

numerics gets complicated.

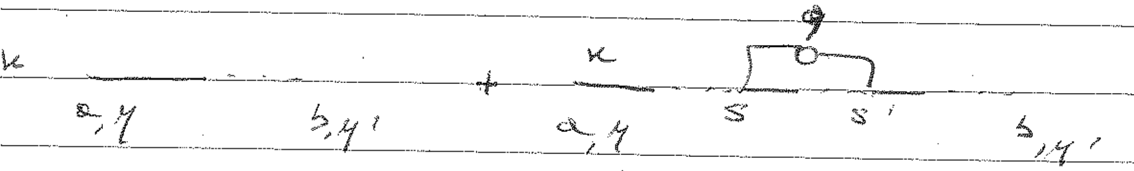
The tree-level vertices are never corrected/renormalized

↳ the improvement over 1-loop standard PT is

not such as to get % level agreement with N-body

Standard PT and the need for resummations

Let's compute the propagator at 1-loop:



$$g_{ab}(\eta-\eta') + 4\underbrace{\int\frac{d^3q}{V(2\pi)^3}\, g_{ac}(\eta-s)\, e^{s}\,\gamma_{cde}(k,q,|k-q|)\, u_d u_g P(q)\, g_{ef}(s-s')\, e^{s'}\,\gamma_{fgh}(|k-q|,q,k)\, g_{hb}(s'-\eta')}_{\text{same integral as in } P_{13}}$$

Now, consider the large-k limit (small scales)

$k \gg q_L$ long

$$\gamma_{abc}(k,q,|k-q|)\,u_b \simeq \delta_{ac}\,\overset{=1}{u_2}\,\frac{\vec{k}\cdot\vec{q}}{q^2}\,\frac{1}{2} + O\left(\left(\frac{q}{k}\right)^0\right)$$

$$\left(\gamma_{abc}(|k-q|,q,k)\,u_b \sim -\delta_{ac}\,u_2\,\frac{\vec{k}\cdot\vec{q}}{q^2}\,\frac{1}{2}\right)$$

"eikonal limit" for the trilinear vertex

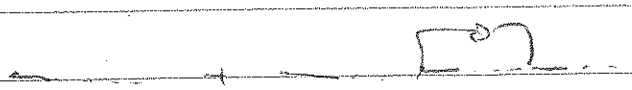
$$\simeq -g_{ab}(\eta-\eta')\underbrace{\int_{\eta'}^{\eta}ds\int_{\eta'}^{s}ds'\, e^{s+s'}}_{\frac{1}{2}(e^{\eta}-e^{\eta'})^2}\int\frac{d^3q}{(2\pi)^3}\,P(q)\,\frac{(q\cdot k)^2}{q^2}$$

$$= -g_{ab}(\eta-\eta')\,\frac{k^2\sigma_v^2}{2}\,(e^{\eta}-e^{\eta'})^2 \qquad \sigma_v^2 = \frac{1}{3}\int\frac{d^3q}{(2\pi)^3}\,\frac{P(q)}{q^2}$$

1-dim velocity dispersion

$(\propto\langle v^i v^i\rangle)$

then, in the large-k limit:


$= g_{ab}(\eta-\eta')\left[1 - \frac{k^2\sigma_v^2}{2}(e^{\eta}-e^{\eta'})^2 + \ldots\right]$

tree — 1-loop — 2-loop

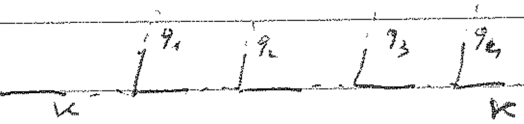
for $k^2 \gtrsim \left(\frac{\sigma_v^2 e^{2\eta}}{2}\right)^{-1}$ 1-loop larger than tree level !!

Using ΛCDM values $\Rightarrow$ $\left(\frac{\sigma_v^2 e^{2\eta}}{2}\right)^{-1} \sim 0.16\ h/\text{Mpc}$ at $z=0$

$\left(\sim \left(6\ \frac{\text{Mpc}}{h}\right)^{-1}\right)$

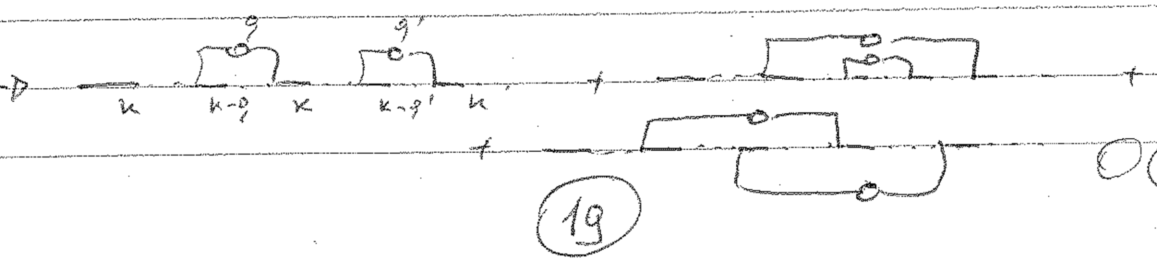
Standard PT breaks down (at least for some quantities) for $k \gtrsim 0.16\ h/\text{Mpc}$ (and likely before)

Two-loop: The leading (in $\frac{k}{q}$) contribution comes from diagrams of this kind



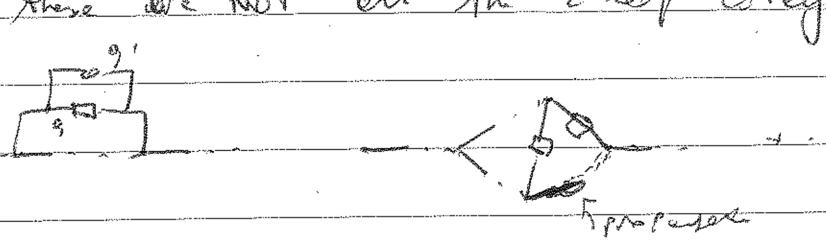
with $k \gg q_i$, since each vertex gives a $\frac{k\cdot q}{q^2}$ factor

then, pair the soft legs in all possible ways

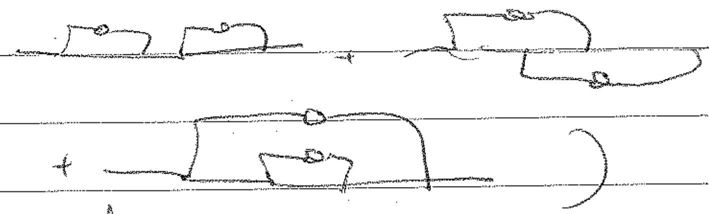


$\rightarrow \ldots + \ldots + \ldots \quad \mathcal{O}\left(\frac{k^4}{q^4}\right)$

Notice: these are NOT all the 2-loop diagrams!



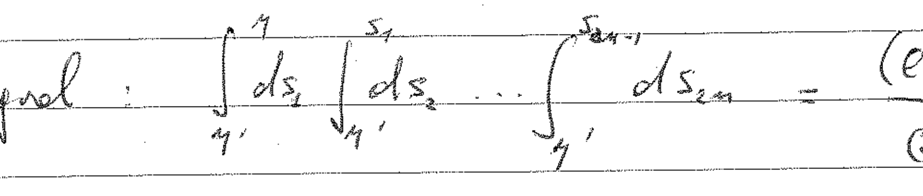
these are subdominant in k

2-loop
the "chain diagrams" ()

give a total contribution of

$$\frac{1}{2}\left(-\frac{k^2\sigma_v^2(e^{\eta}-e^{\eta'})^2}{2}\right)^2$$

At any n-loop, the contribution can be obtained from



$2n$ legs

time integral: $\displaystyle\int_{\eta'}^{\eta} ds_1 \int_{\eta'}^{s_1} ds_2 \cdots \int_{\eta'}^{s_{2n-1}} ds_{2n} = \frac{(e^{\eta}-e^{\eta'})^{2n}}{(2n)!}$

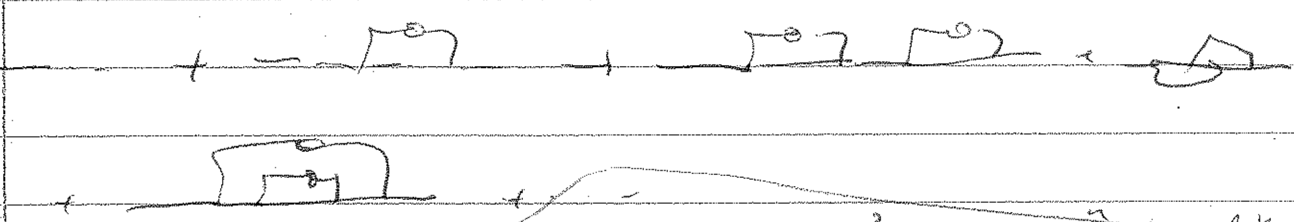
$\times$ # of pairings $(2n-1)!!$

each one of the pairings gives a $-k^2\sigma_v^2$ factor $\rightarrow (-k^2\sigma_v^2)^n$

$\Rightarrow$ n-loop contribution from chain diagrams:

$$\frac{(2n-1)!!}{(2n)!}\left(-k^2\sigma_v^2\left(e^\eta-e^{\eta'}\right)^2\right)^n = \frac{1}{n!}\left(\frac{-k^2\sigma_v^2\left(e^\eta-e^{\eta'}\right)^2}{2}\right)^n$$

$\rightarrow$ the series of chain diagrams can be resummed at all orders! (Crocce - Scoccimarro 2006)



$$= \boxed{g_{ab}(\eta-\eta')\, e^{-\frac{k^2\sigma_v^2\left(e^\eta-e^{\eta'}\right)^2}{2}} \equiv G^{eik}_{ab}(k;\eta,\eta')}$$

"eikonal"

Notice: - scale-dependent
- time-translation invariance lost } already true at 1-loop
- Gaussian damping in the UV!!

$\rightarrow$ The UV behavior of the resummed propagator is completely different than that of any finite order truncation $(O(k^{2n}))$

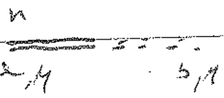
→ RPT: Renormalized Perturbation Theory:

Use as tree level propagator not the linear one but one that has the eikonal limit at large k.

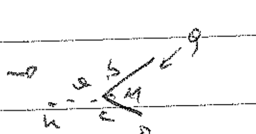
$$g_{ab}(\eta-\eta') \longrightarrow G^{cs}_{ab}(k;\eta,\eta') \begin{cases} \rightarrow g_{ab} + \Delta g^{1\text{-loop}}_{ab} & k\to 0 \\ \rightarrow G^{eik}_{ab}(k;\eta,\eta') & k\to\infty \end{cases}$$

and an interpolation in between.

New Feynman rules:


$G^{cs}_{ab}(k;\eta,\eta')$


$G^{cs}_{ac}(k;\eta,-\infty)\, u_c u_d\, P(k)\, G^{cs}_{bd}(k;\eta,-\infty)$

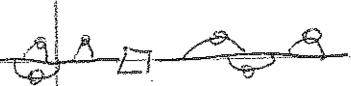

$: e^{\eta}\gamma_{abc}(k,q,|k+q|)$

The nonlinear PS is then given by

$$P^{NL}_{ab}(k;\eta) = G^{CS}_{ac}(k;\eta,-\infty)\, G^{CS}_{bd}(k;\eta,-\infty)\, u_c u_d P(k) +$$

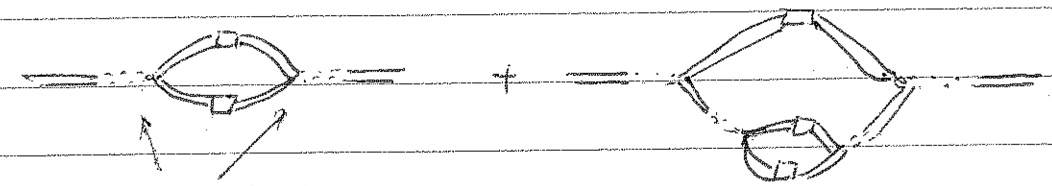
$$+ P^{MC}_{ab}(k;\eta) \quad \leftarrow \text{"Mode-coupling" term}$$

[it comes from the exact expression $P_{ab} = G P G + P^{MC}$ with G the full prop. and $P^{MC} = \dots$]

 proportional to the linear PS

$\tilde{G}^2 P(k)$

P^{MC}: (CS '08)



Tree-level vertices

NO VERTEX REN. AT THIS LEVEL.

Effects of the mode-coupling term at this level of approximation: $\lesssim 0.5\%$ shift of the BAO peak position at $z=0$. (Crocce Scoccimarro '08)

* Notice: in order to reproduce the N-body data on the propagator, CS had to introduce a correction to the exponential decay of the $G^{CS} \sim e^{-\frac{k^2 \sigma_v^2 e^{2\eta}}{2}}$ in the form $\sigma_v^2 \to \sigma_v^2 \alpha^2(z)$ with

$$\alpha^2(z) = \frac{\int d^3q \, \frac{P_{NL}(q,z)}{q^2}}{\int d^3q \, \frac{P_{lin}(q,z)}{q^2}}$$

with P_{NL} taken from halofit ($\alpha^2(z=0) \sim 1.05$)

Questions on RPT:

The CS-propagator is obtained after resummation of soft modes: but what separates soft from hard?

$$\left(\int^{\infty} \frac{d^3 q}{(2\pi)^3} \frac{P(q)}{q^2} \right)$$

The physical effects which is "resummed" should be velocity large scale flows (bulk flows). But this should have NO impact on equal time correlators (in the uniform limit)

should be $P^{Res}(k;\eta) = P^{lin}(k;\eta)$ at tree-level in RPT

but it is: $G^2(k) P^{lin} \sim e^{-k^2\sigma^2 e^{2\eta}} P^{lin}$!!

what's going on?!

It is useful to introduce an explicit separation of scale, say $\bar{k}$, such that modes with $k \ll \bar{k}$ are called LONG and modes $k \gg \bar{k}$ are called SHORT.

$$\varphi_a(k,\eta) = \varphi_a^L(k,\eta) + \varphi_a^S(k,\eta)$$

we are interested in the short modes

$$\rightarrow \left(\delta_{ab}\partial_\eta + \Omega_{ab}\right)\varphi_b^S(k,\eta) = e^\eta \int \gamma_{abc}\left(\varphi_b^S\varphi_c^S + \varphi_b^L\varphi_c^S + \varphi_b^S\varphi_c^L\right)$$

The LONG MODES are assumed to be LINEAR (AND GAUSSIAN)

Consider a LONG VELOCITY MODE, v_L^i and define

$$\vec{D}(\vec{x},\tau) = \int_{\tau_{in}}^{\tau} d\tau'\, \vec{v}_L(\vec{x},\tau') \qquad \text{Total shift}$$

$$= \int_{\eta_{in}}^{\eta} d\eta' \frac{\vec{v}_L(\vec{x},\eta')}{\mathcal{H} f_+} \qquad \eta = \ln D_+$$

$$= e^\eta \left.\frac{\vec{v}_L(\vec{x},\eta_{in})}{\mathcal{H} f_+(\eta_{in})}\right|_{\eta_{in}\to-\infty} =$$

const in linear th.

$$\vec{D}(0,\eta) \equiv \vec{D}(\eta) = i\int \frac{d^3q}{(2\pi)^3} e^\eta \frac{\vec{q}}{q^2}\varphi_2^L(q) \qquad \theta(q) = i\, q^i v^i$$

$\vec{D}(x)$ const in $\vec{x}$

in the ~~...~~ limit of this a uniform velocity shift

"Generalized Galilean transformation".

Shift the short fields $\varphi^S(\vec{x}) \to \bar{\varphi}^S(\vec{x}') = \varphi^S(x - \vec{D}(\tau))$

in Fourier space: $\bar{\varphi}_a^S(\vec{k},\eta) = e^{i\vec{k}\cdot\vec{D}(\eta)}\varphi_a^S(\vec{k},\eta)$

the equation for $\bar{\varphi}_a^S$ fields is $\longrightarrow$

$$(\delta_{ab}\partial_\eta + \Omega_{ab})\,\bar\varphi^s_b(k,\eta) = e^\eta \gamma_{abc}\,\bar\varphi^s_b\,\bar\varphi^s_c \;+$$

γ' - peak

$$+\Big\{ e^\eta \int \frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\Big[\delta(k-p-q)\,\gamma_{abc}(k,p,q) - \delta(k-p)\,\frac{1}{2}\frac{\vec k\cdot\vec q}{q^2}\,\delta_{ab}\,\delta_{c2}\Big]\times$$

$$\times\,\bar\varphi^s_b(p,\eta)\,\varphi^L_c(q,\eta)\,e^{-i(\vec p-\vec k)\cdot\vec D(\eta)}$$

$$+\; e^\eta \int \ldots \Big[\delta(k-p-q)\,\gamma_{abc}(k,p,q) - \delta(k-q)\,\frac{1}{2}\frac{\vec k\cdot\vec p}{p^2}\,\delta_{ac}\,\delta_{b2}\Big]$$

$$e^{-i(\vec q-\vec k)\cdot\vec D(\eta)}\,\varphi^L_b(p)\,\bar\varphi^s_c(q,\eta)\;\Big]\;\Big\}$$

Notice that in the full eikonal limit, $\gamma' = \gamma_{abc}$,

the field $\bar\varphi^s$ is completely decoupled from φ^L.

In this limit $\langle \varphi^s_a(k,\eta)\,\varphi^s_b(k',\eta')\rangle = \langle e^{-i(\vec k\cdot D(\eta)+\vec k'\cdot D(\eta'))}\,\bar\varphi^s_a(k,\eta)\,\bar\varphi^s_b(k',\eta')\rangle$

The correlator for the original fields reads $\to$

$$= \langle e^{-i(k\cdot D + k'\cdot D(\eta'))}\rangle\,\langle\bar\varphi^s_a(k,\eta)\,\bar\varphi^s_b(k',\eta')\rangle$$

$$\delta(k+k') \;\to\; e^{-\frac{\langle (k\cdot(D(\eta)-D(\eta')))^2\rangle}{2}}\,\langle\bar\varphi^s_a(k,\eta)\,\bar\varphi^s_b(k',\eta')\rangle$$

$$\langle (k\cdot(D(\eta)-D(\eta')))^2\rangle = k^2\sigma_v^2\,(e^\eta - e^{\eta'})^2$$

$$\Rightarrow P^{s}_{ab}(k;\eta,\eta') = e^{-\frac{k^2\sigma^2}{2}(e^{\eta}-e^{\eta'})^2}\,\tilde{P}^{s}_{ab}(k;\eta,\eta')$$

So, not only $G^{s}_{ab}(k;\eta,\eta')$ has ~~the~~ $e^{-\frac{k^2\sigma^2}{2}(e^{\eta}-e^{\eta'})^2}$ factor

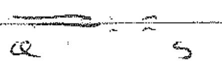
but also the PS. $P^{s}_{ab}(k;\eta,\eta) = \tilde{P}^{s}_{ab}(k;\eta,\eta)$.

Totally independent on the long modes IN THE FULLY EIKONAL LIMIT. ~~DIFFERENT~~

the PS in the eikonal limit is not that of RPT

$\Rightarrow$ eRPT (Anselmi, MP, also Peloso, MP)

New Feynman rules:

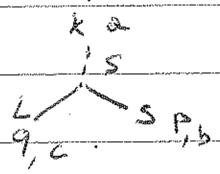


$$= G^{eRPT}_{ab}(k;\eta,\eta') = G^{eik}_{ab}(k;\eta,\eta') = e^{-\frac{k^2\sigma^2}{2}(e^{\eta}-e^{\eta'})^2}\, g_{ab}(\eta-\eta')$$

DO INTERFACE

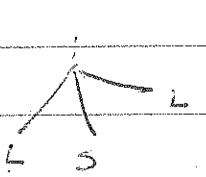
$$= P^{eRPT}_{ab}(k;\eta,\eta') = e^{-\frac{k^2\sigma^2}{2}(e^{\eta}-e^{\eta'})^2}\, u_a u_b P(k)$$

$$= e^{\eta}\gamma_{abc}\,\delta(\vec{k}-\vec{p}-\vec{q})$$



$$= e^{\eta}\left(\gamma_{abc}\delta(\vec{k}-\vec{p}-\vec{q}) - \delta(k-p)\,\delta_{ab}\,\delta_{c2}\,\frac{\vec{k}\cdot\vec{q}}{q^2}\,\frac{1}{2}\right)\times$$

but also from the phase $e^{-i(\vec{p}-\vec{k})\cdot\vec{D}(\eta)}$



$\Rightarrow$ TO BE FULLY WORKED OUT E.G. IN A 1-LOOP COMPUTATION

~~any~~ all orders, the $\bar{k}$-dependence must vanish.
However, at finite orders ~~#~~ there is a residual k-dep.

RPT consistently takes into account the effect of
bulk flows at any finite order, RPT does not.

RPT is manifestly Galilean invariant.
In RPT one considers a set of diagrams that is NOT GI
[1-loop example]
the effect of the SSL vertices is to remove
an eikonal vertex and to replace it with an
exact "tree-level" one, with the correct momentum
dependence. It is crucial, since there is nothing
providing such an efficient separation of scales as to
separate short from long modes

. The SSS vertex accounts for mode-mode
couplings between short modes (with gaussian damping)

The Γ-expansion (multi-point propagator expansion)

It is a very general result that can be applied to any nonlinear field transformation (that is, not only for gravitational dynamics.)

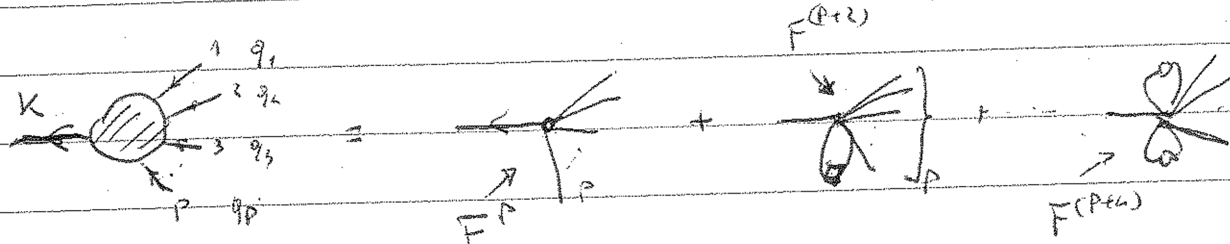
Consider a field $\varphi(\vec{k};\eta)$ and expand it in terms of the initial Gaussian fields $\varphi^{in}(\vec{k})$:

$$\varphi(\vec{k},\eta) = \sum_n I_{k,q_1\cdots q_n} F^{n}(q_1,\dots q_n;\eta)\,\varphi^{in}(q_1)\cdots\varphi^{in}(q_n)$$

Define the p-point propagator as

$$\frac{1}{p!}\left\langle \frac{\delta\varphi(\vec{k},\eta)}{\delta\varphi^{in}(q_1)\dots\delta\varphi^{in}(q_p)}\right\rangle = \delta_D\Big(k-\sum_i^{p}\vec{q}_i\Big)\,\Gamma^{(p)}(q_1\dots q_p;\eta)$$

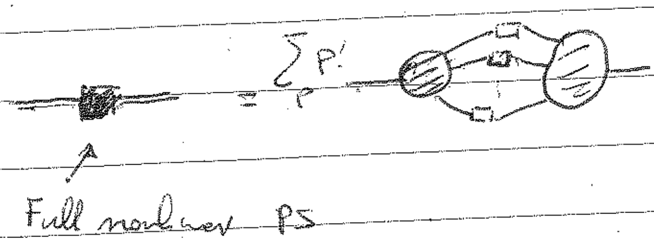
ensemble average



y result, (Bernardeau et al 2012):

$$\langle \varphi(\vec{k},\eta)\,\varphi(\vec{k}',\eta)\rangle = \overbrace{(2\pi)^3\delta_D(k+k')}\sum_{1}^{\infty}{}_{p}\; p!\; \int_{k,q_1\dots q_p} \times$$

$$\left[\Gamma^{(p)}(q_1,\dots,q_p;\eta)\right]^2 P(q_1)\cdots P(q_p)$$

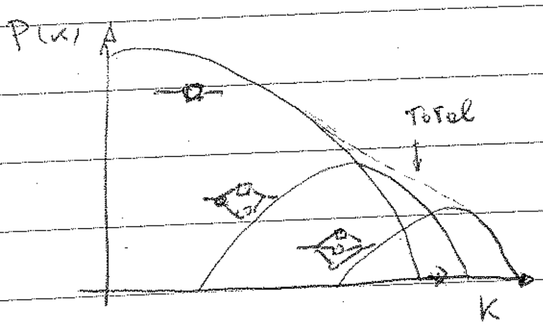


- EXACT RESULT
- ~~NON~~ EXTENSION TO NON GAUSSIAN IS POSSIBLE

IT represents an ~~the~~ alternative expansion w.r.t. SPT:

~~I~~nstead of the kernels $F^{(n)}$, the multipoint propagators

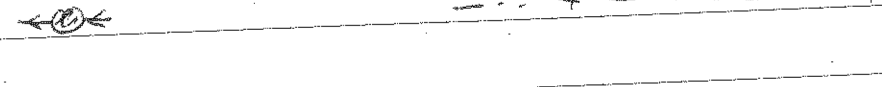
$\Gamma^{(n)}$. Notice that each term in the sum is positive:



$$p=1 \;\longrightarrow\; \left\langle \frac{\delta\varphi_a(k,\eta)}{\delta\varphi^{in}_b(k')}\right\rangle = \delta_D(k-k')\,\Gamma^{(1)\,b}_{a}(k;\eta)$$

this is exactly the nonlinear propagator:

$$\Gamma^{(1)\,b}_{a}(k;\eta) = G_{ab}(k;\eta,\eta_{in})$$



all SPT contributions (NOT ONLY CHAIN)

In practical computations $\Gamma_a^{(1)b}$ is approximated with the "CS-like" propagator:

$$G^{CS}_{ab}(k;\eta,\eta') = \left[g_{ab}(\eta-\eta') + \Delta g^{1\text{-loop}}_{ab}(k;\eta,\eta') + \frac{1}{2}k^2\sigma_v^2(e^{\eta}-e^{\eta'})^2 \right] \times \exp\left(-\frac{1}{2}k^2\sigma_v^2(e^{\eta}-e^{\eta'})^2\right)$$

$$\xrightarrow{k\to\infty} g_{ab}(\eta-\eta')\exp\left(-\frac{1}{2}k^2\sigma_v^2(e^{\eta}-e^{\eta'})^2\right)$$

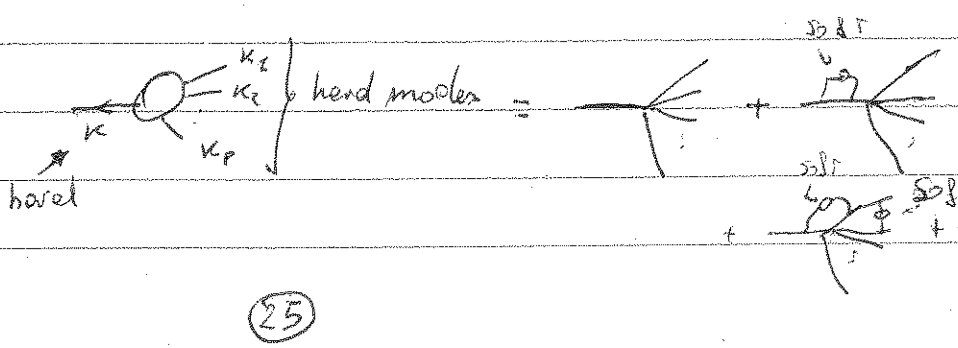
$$\xrightarrow{k\to 0} g_{ab}(\eta-\eta') + \Delta g^{1\text{-loop}}_{ab}(k;\eta,\eta')$$

$$\Delta g^{1\text{-loop}} \xrightarrow[k\to\infty]{} -\frac{1}{2}k^2\sigma_v^2(e^{\eta}-e^{\eta'})^2$$

Can be generalized for the p-point propagator:

$$\Gamma_a^{reg,b_1\dots b_p}(k_1,\dots k_p;\eta,\eta') = \left[\Gamma_a^{tree,b_1\dots b_p} + \delta\Gamma_a^{1\text{-loop},b_1\dots b_p} + \frac{1}{2}k^2\sigma_v^2(e^{\eta}-e^{\eta'})^2\,\Gamma_a^{tree,b_1\dots b_p}\right]\exp\left(-\frac{k^2\sigma_v^2(e^{\eta}-e^{\eta'})^2}{2}\right)$$

At large k, the exponential is the result of summing soft corrections:



Available codes compute the sum up to $p=3$:

$p=1$ up to two-loop orders
$p=2$ " " 1-loop
$p=3$ at tree level

~~[illegible]~~

Notice: the truncation at finite p breaks Galilean invariance! (mode-mode coupling effects of soft modes only up to $p=3$, "propagator" corrects at all orders)

2 available tools:

- MPTBREEZE: Crocce ~~Scoccimarro~~ Bernardeau
  1207.1465
  code from MAIA.ICE.CAT/CROCCE/MPTBREEZE

RegPT: Taruya Bernardeau Nishimichi ~~[illegible]~~ Codis
1208.1191

Google "TARUYA REGPT CODE"

RegPT is "2-loop", MPTBreeze "1-loop" for $p=1$,

p=2,3 are the same, ~ % accuracy in the BAO range
rapidly degrading at smaller scales

Time-flow equations:

One example is the TRG (see these notes p.14)

A more efficient method, in terms of accuracy and computing time is presented in Anselmi, Pietroni 1205.2235 (see also 1011.4477)

First, let's consider the propagator.

The full nonlinear propagator ~~solves the equation~~ G can be expressed as

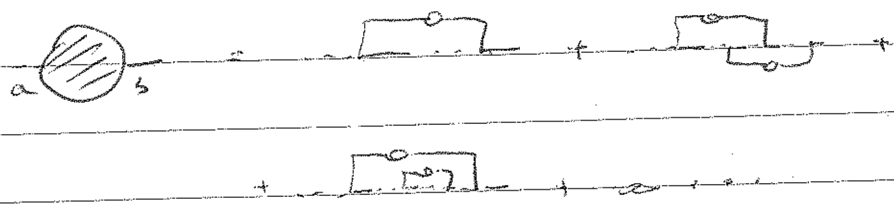
$$G_{ab}(k;\eta,\eta') = [g^{-1} - \Sigma]^{-1}_{ab}(k;\eta,\eta') =$$

$$= g_{ab}(\eta-\eta') + \int ds\, ds'\, g_{ac}(\eta-s)\,\Sigma_{cd}(k;s,s')\,g_{db}(s'-\eta') +$$

$$+ \int g\Sigma g\Sigma g + \dots$$

$$= g_{ab}(\eta-\eta') + \int^{\eta} ds \int^{s} ds'\, g_{ac}(\eta-s)\,\Sigma_{cd}(k;s,s')\,G_{db}(k;s',\eta') \qquad (*)$$

...ere $\Sigma_{ab}(k;y,y')$ is the 1PI (1 particle irreducible)

function 

Computing Σ is then equivalent to computing G.

...ke the y-derivative of (*)

$$\partial_y G_{ab}(k;y,y') = f_{ab}\,\delta_D(y-y') - \Omega_{ac}G_{cb}(k;y,y') + \Delta G_{ab}(k;y,y') \qquad (**)$$

where $\Delta G_{ab}(k;y,y') = \int_{y'}^{y} ds'\, \Sigma_{ac}(k;y,s')\, G_{cb}(k;s',y')$

(**) is completely equivalent to (*)

In the large-k limit one can show that

$$\Delta G_{ab}(k;y,y') \to -k^2\sigma_v^2\, e^{y}(e^{y}-e^{y'})\, G_{ab}(k;y,y')$$

$\to$ (**) can be integrated to give $G_{ab}(k;y,y') = G^{eik}_{ab}(k;y,y')$

In the small-k limit $\Delta G_{ab}(k;y,y') = \int_{y'}^{y} ds'\, \Sigma^{1-loop}_{ac}(k,y,s')\, g_{cb}(s'-y')$

$$\Rightarrow \Delta G_{ab}(k;y,y')\,u_b \xrightarrow[\text{small } k]{} g_{ab}(y-y')\,u_b \int_{y'}^{y} ds'\, \Sigma^{1-loop}_{ac}(k,y,s')\,u_c$$

$$\xrightarrow[\text{large } k]{} G_{ab}(k;y,y')\,u_b \int_{y'}^{y} ds'\, \Sigma^{1-loop}_{ac}(k;y,s')\,u_c$$

the same factorized form holds both at small k (modulo 2-loop terms) and at large-k.

Integrating the $(\alpha\alpha)$ equation, using the factorized form for Δ_{ab}, we get a propagator interpolating between the 1-loop one and the eikonal one.

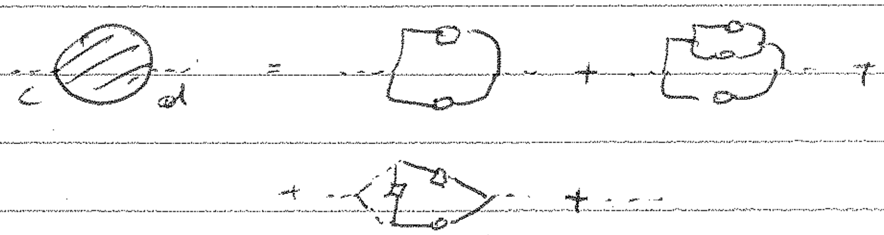
We only need a 1-loop computation, $\Sigma_{ab}^{1\text{-loop}}$, to resum infinite loops.

A similar scheme can be employed for the PS.

We start from the exact relation:

$$P_{ab}(k;\eta) = G_{ac}(k;\eta,\eta_{in})\, G_{bd}(k;\eta,\eta_{in})\, P(k)\, u_c u_d \; +$$
$$+ \int_{\eta_{in}}^{\eta} ds \int_{\eta_{in}}^{\eta} ds'\; G_{ac}(k;\eta,s)\, G_{bd}(k;\eta,s')\, \Phi_{cd}(k;s,s')$$

where, ~~apart~~ a new 1PI function appears, Φ_{cd}



We can obtain a time evolution equation for P

ordering by η and ~~eqs~~ treating the propagator as

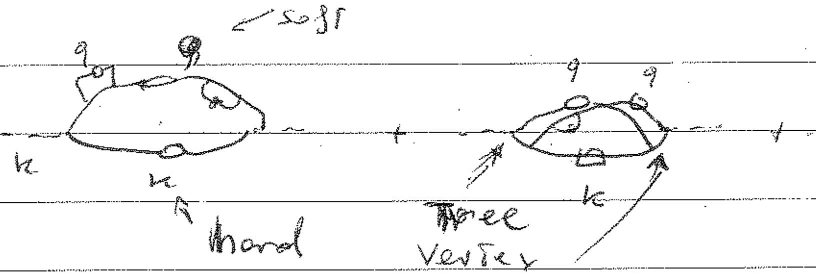
before. As for Φ_{ab} we can use the 1-loop

expression $\longrightarrow$ ~2% accuracy in all the BAO region down

to $z=0$.

We can do better, by summing all contributions of

the form



$\hookrightarrow$ ~~accuracy~~ few % accuracy up to $k \sim 0.3\, h/\mathrm{Mpc}$

at $z > 0.5$!!

• Mathematica code ~~out~~ available.

very fast: ~~only~~ 1 loop INTEGRALS + 1 time

integration.

COARSE GRAINED PT / EFFECTIVE FIELD THEORY APPROACH

As we have seen SPT fails in the UV: higher loops become less and less UV convergent, nonlinearities grow at large k and the expansion breaks down, the single stream approximation will certainly break down inside virialised objects, but its effects would presumably influence also larger scales.

The general idea of these approaches is to gain information on the UV sector of the theory from N-body simulation (or from observation) and to use it as ~~as~~ source terms for ~~fluid~~ imperfect fluid equation describing the ~~IR sec~~ large scales, which can then be treated perturbatively.

..., as a starting point, we introduce a smoothing scale L, and want to derive the fluid equations for fields smoothed ~~of~~ over such a scale.

We start from the "microscopic" Vlasov equation

$$\left(\frac{\partial}{\partial \tau} + \frac{p^i}{a m}\frac{\partial}{\partial x^i} - a m \nabla^i \phi \frac{\partial}{\partial p^i}\right) f(\vec{x}, \vec{p}, \tau) = 0$$

and consider the smoothed distribution function

$$\bar{f}(\vec{x}, \vec{p}, \tau) = \int d^3y \, W\left[\frac{y}{L}\right] f(\vec{x}-\vec{y}, \vec{p}, \tau)$$

with W a filter function such that $\int d^3y \, W\left[\frac{y}{L}\right] = 1$

$$\left(\text{e.g. } W\left[\frac{y}{L}\right] = \frac{1}{L^3}\frac{1}{(2\pi)^{3/2}} e^{-x^2/2L^2} \xrightarrow{F.T.} \tilde{W}[kL] = e^{-\frac{k^2L^2}{2}}\right)$$

if we apply the filter to the Vlasov eq. we get

$$\left(\frac{\partial}{\partial \tau} + \frac{p^i}{a m}\frac{\partial}{\partial x^i} - a m \nabla^i \bar{\phi} \frac{\partial}{\partial p^i}\right) \bar{f}(\vec{x}, \vec{p}, \tau) = - K[f](x, p, \tau)$$

smoothed potential

$$\nabla^2 \bar{\phi} = \frac{3}{2} \mathcal{H}^2 \Omega_m \bar{\delta}$$

where the "pseudo-collisional" term is given by

$$K[f](x,p,\tau) = -\langle a m \nabla^i \phi \frac{\partial}{\partial p^i} f(x,p,\tau)\rangle_L +$$

$$+ \; a m \nabla^i \bar{\phi} \frac{\partial}{\partial p^i} \bar{f}(x,p,\tau)$$

with $\langle \, \rangle_L$ we indicate the volume average, i.e.

$$\langle g(\vec{x})\rangle_L \equiv \bar{g}(\vec{x}) = \int d^3y \, W(y/L) \, g(\vec{x}-\vec{y}) .$$

The moments of $\bar{f}$ are related to those of f by

$$\bar{n}(\vec{x},\tau) = \int d^3p \, \bar{f}(\vec{x},\vec{p},\tau) = \langle n\rangle(\vec{x},\tau)$$

$$\bar{v}^i(\vec{x},\tau) = \frac{1}{\bar{n}}\int d^3p \frac{p^i}{am} \bar{f} = \frac{1}{1+\bar{\delta}} \langle (1+\delta) v^i\rangle$$

$$\bar{\sigma}^{ij} = \frac{1}{\bar{n}(x,\tau)} \int d^3p \frac{p^i p^j}{a^2 m^2} \bar{f} - \bar{v}^i \bar{v}^j =$$

$$= \frac{1}{1+\bar{\delta}} \langle (1+\delta) v^i v^j + \sigma^{ij} \rangle - \bar{v}^i \bar{v}^j$$

Notice that $\bar{\sigma}^{ij} \neq 0$ even if $\sigma^{ij} = 0$, that is, velocity dispersion is generated by coarse graining even if the single stream regime holds microscopically ow, the equations for the moments are

$$\begin{cases} \frac{\partial}{\partial\tau}\bar{\delta} + \frac{\partial}{\partial x^i}\left((1+\bar{\delta})\bar{v}^i\right) = 0 \\ \frac{\partial}{\partial\tau}\bar{v}^i + \mathcal{H}\bar{v}^i + \bar{v}^k \frac{\partial}{\partial x^k}\bar{v}^i = -\nabla^i\bar{\phi} - \bar{J}^i \\ \nabla^2\bar{\phi} = \frac{3}{2}\mathcal{H}^2\Omega_m\bar{\delta} \end{cases}$$

where the source term at the RHS of the Euler

equation is:

$$\bar{J}^i \equiv J_1^i + J_\sigma^i$$

$$J_1^i \equiv \frac{1}{1+\bar{\delta}}\langle(1+\delta)\nabla^i\phi\rangle_L - \nabla^i\bar{\phi}$$ Force fluctuations inside the cg-volume

$$J_\sigma^i \equiv \frac{1}{1+\bar{\delta}}\frac{\partial}{\partial x^k}\left((1+\bar{\delta})\bar{\sigma}^{ik}\right)$$ velocity dispersion

Notice that the set of equation is exact, i.e. no truncation, single stream approximation, etc. has been assumed.

The system is not closed: need external

input for $\mathcal{J}^{i}$ (see later).

We can use the compact notation: $\bar{\varphi}_a = e^{-\eta}\begin{pmatrix} \delta \\ -\frac{\bar{\theta}}{\mathcal{H}f} \end{pmatrix}$

$$h_a = -i\,\frac{k^j J^j}{\mathcal{H}^2 f^2}\, e^{-\eta}\,\delta_{a2}$$ to get the equation

for the smoothed fields:

$$(\delta_{ab}\,\partial_\eta + \Omega_{ab})\,\bar{\varphi}_b = e^{\eta}\, I_{k q_1 q_2}\, \gamma_{abc}\, \bar{\varphi}_b \bar{\varphi}_c$$
$$- h_a(k,\eta) + \text{"vorticity"}$$

where the "vorticity" terms come from the fact that even if the microscopic velocity has zero vorticity, it gets generated by coarse graining

i.e. $\bar{v}_x^{\,l} = i\,\varepsilon_{lmn}\, k^m \bar{v}^n \neq 0$ even if $\varepsilon_{lmn} k^m v^n = 0$

$$\left[\; i\,\varepsilon_{ijl}\,\frac{k_j v_x^{\,l}}{k^2} = -i\, I_{k,q,p}\left(\frac{q^i}{q^2} - \frac{k^i\, q\cdot k}{k^2 q^2}\right)\right)\Big(\langle \delta(p)\theta(q)\rangle - \bar{\delta}(p)\bar{\theta}(q)\Big) + O(\delta^2 v) \;\right]$$

the vorticity terms are subleading and can be treated perturbatively. We will ignore them for sake of simplicity, ut they are taken into account in the computation.

To solve the equation ~~perturbat~~ we first define the field $\tilde{\varphi}_a = \bar{\varphi}_a + \int^{\eta} ds\, g_{ab}(\eta - s) h_b(s)$

$$\Rightarrow (\delta_{ab}\partial_\eta + \Omega_{ab})\tilde{\varphi}_b = e^{\eta} I_{k,q_1q_2} \gamma_{abc}(\tilde{\varphi}_b - (gh)_b)(\tilde{\varphi}_c - (gh)_c)$$

$$+ \text{"vorticity"}$$

and then expand perturbatively in γ and in h.

The power spectrum is then given by

$$\langle \bar{\varphi}_a \bar{\varphi}_b \rangle = \langle \tilde{\varphi}_a \tilde{\varphi}_b \rangle - \int^{\eta} ds \left[g_{ac}(\eta - s) \langle h_c(s) \tilde{\varphi}_b(\eta) \rangle + a \leftrightarrow b \right]$$

$$+ \int^{\eta} ds \int^{\eta} ds'\, g_{ac}(\eta - s) g_{bd}(\eta - s') \langle h_c(s) h_d(s') \rangle$$

The $\langle \tilde{\varphi} \tilde{\varphi} \rangle$ term is computed perturbatively, it contains loop integrals up to $q \lesssim \ell^{-1}$, well behaved.

The source term correlators $\langle h \tilde{\varphi} \rangle$ and $\langle hh \rangle$

contain information on all physics at scales $q > L^{-1}$,

and can be obtained from simulations

Notice that the source terms contain information

on: - all the nonlinearities at $q > L^{-1}$, even if the

single stream approx holds microscopically,

(SSA)

- the breaking of the single stream approx

at a microscopic level.

In practice, it is difficult to single out the

pure SSA breaking effect from simulation (see Pueblas,

Scoccimarro 2009). In this approach we do not need

to do so, if we work at finite L.

To proceed, there are two approaches:

1) measure the $\langle h\varphi \rangle$ $\langle hh \rangle$ correlators directly

from N-body simulations → CGPT (Pietroni et al 2012)
Peloso et al to appear

2) Express the source ~~term~~ in terms of the smoothed fields, and read the coefficients from N-body simulation, either directly (i.e. through the correlators) or by fitting the coefficients appearing in the final expression for the PS. → EFT
(Baumann et al 2012
Carrasco et al 2012)

$$J^i = \frac{1}{a\bar{\rho}} \partial_j \tau^{ij}$$

effective stress tensor

bulk viscosity

$$\tau^{ij} = \bar{p}\,\delta^{ij} + \bar{\rho}\left[c_s^2 \bar{\delta}\,\delta^{ij} - \frac{c_{bv}^2}{\mathcal{H}}\,\delta^{ij}\partial_k \bar{v}^k - \frac{3}{4}\frac{c_{sv}^2}{\mathcal{H}}\left(\partial^j \bar{v}^i + \partial^i \bar{v}^j - \frac{2}{3}\delta^{ij}\partial_k \bar{v}^k\right)\right]$$

speed of sound

shear viscosity

$$+ A\,\tau^{ij} + \ldots$$

stocastic term

... $\longrightarrow$ 1-loop EFT result for the PS

$$P_{EFT\text{-}1loop} = P^{1\text{-}loop}_{SPT} - \underbrace{2(2\pi)^2 \bar{c}_s^2}_{\substack{\text{combination} \\ \text{of } c_s^2,\, c_{bv}^2 + c_{sv}^2}} \frac{k^2}{k_{NL}^2} \underbrace{P(k)}_{\text{linear PS}}$$

$\bar{c}_s^2$ is fitted to the Nonlinear PS at some k,

the prediction is then the nonlinear PS at different

k's.

Open questions:

- COSMOLOGY DEPENDENCE:

if the correlators / c_s^2 depend too strongly on

the cosmology, then one needs to run a simulation

for each different cosmology $\rightarrow$ No advantage.

- Time dependence

- higher orders

- Bispectrum...

Bias

[Up] to now, we have ~~d~~ discussed only the first [st]age of the problem, namely the evolution from [li]near to nonlinear of the MATTER density field

$$\delta(\vec{x}, \tau_{in}) \rightarrow \delta(\vec{x}, \tau)$$

[Th]e next step is to relate the matter density field [to] the galaxy density field (and the next one, to [go] to redshift space, see later).

[Th]is is usually done in two steps:

$$\underset{\text{matter}}{\delta(\vec{x}, \tau)} \longrightarrow \underset{\text{halo}}{\delta_h(\vec{x}, \tau)} \xrightarrow[\substack{\text{galaxy formation} \\ \text{baryonic physics} \\ \text{nonlocal, non-deterministic}}]{} \underset{\text{galaxy}}{\delta_g(\vec{x}, \tau)}$$

[W]e will concentrate on the relation between δ_h and δ_g

[(]DARK MATTER ONLY SYSTEM).

The halo bias was traditionally ~~assumed~~ assumed to be a

local and deterministic process (Fry Gaztanaga '93), that is

$$\delta_h(\vec{x},z) = \mathcal{F}[\delta(\vec{x},z)] = \sum_n \frac{b_n}{n!} \delta^n(\vec{x},z)$$

~~local~~ δ_h in $\vec{x}$ depends only on δ in $\vec{x}$. b_n: bias params

- On large scales $\delta_h(\vec{x},z) = b_1 \delta(\vec{x},z)$ shown to be a good approximation
- On scales larger than $\mathcal{O}(10\,\mathrm{Mpc})$ also the ~~[illegible]~~ stochasticity ~~of the~~ is also negligible (basically shot noise, which decreases by taking larger scales)

More recently (McDonald, Roy '09, Chan Scoccimarro Sheth '12, Assassi et al '14) it has been realized that

the relation $\delta_h = \mathcal{F}[\delta]$ is not accurate,

as δ_h does not depend on δ only: NON-local bias

~~[illegible]~~ To see it, consider a simplified situation

in which halo comoving number is conserved

(i.e. no merging) & it can be modeled by source terms)

$$\Rightarrow \begin{cases} \dfrac{\partial \delta}{\partial \tau} + \vec{\nabla}\left((1+\delta)\vec{v}\right) = 0 \\ \dfrac{\partial}{\partial \tau} v^k + \mathcal{H} v^k + v^j \dfrac{\partial}{\partial x^j} v^k = -\nabla^k \phi \\ \dfrac{\partial \delta_h}{\partial \tau} + \vec{\nabla}\left((1+\delta_h)\vec{v}_h\right) = 0 \\ \dfrac{\partial v_h^k}{\partial \tau} + \mathcal{H} v_h^k + v_h^j \dfrac{\partial}{\partial x^j} v_h^k = -\nabla^k \phi \\ \nabla^2 \phi = \dfrac{3}{2}\mathcal{H}^2 \Omega_m \delta \end{cases}$$

same ϕ, different v

The system is invariant under the boost symmetry (Generalized Galilean invariance) (Notice: also source terms would be INVARIANT!)

$$\begin{cases} \vec{x} \to \vec{x} - T(\tau)\vec{u} = \vec{x}' \\ \tau \to \tau = \tau' \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} \delta'(\vec{x},\tau) = \delta(\vec{x}',\tau') \\ \delta_h'(\vec{x},\tau) = \delta_h(\vec{x}',\tau) \\ \vec{v}'(\vec{x},\tau) = \vec{v}(\vec{x}',\tau') + \dot{T}\vec{u} \\ \vec{v}_h'(\ \ ) = \vec{v}_h \qquad + \dot{T}\vec{u} \\ \phi_0'(\vec{x},\tau) = \phi(\vec{x}',\tau') - [\mathcal{H}\dot{T} + \ddot{T}]\,\vec{u}\cdot\vec{x} \end{cases}$$

Notice that δ_h is a scalar under these transformations

In general, it will then depend on all the other scalars which can be built.

One can use as fundamental fields ϕ and ϕ_v,

defined as $\nabla^2\phi_v \equiv \theta = \vec{\nabla}\cdot\vec{v}$

Under boosts, $\phi_v' \Rightarrow \phi_v + \dot{T}\vec{u}\cdot\vec{x}$

Scalars can be constructed from ϕ and ϕ_v

using $\nabla_i\nabla_j\phi$, $\nabla_i\nabla_j\phi_v$. ($\phi_{(v)}$ and $\nabla_i\phi_{(v)}$ cannot

have physical effects)

1^{st} order: $\phi^{(1)} = \phi_v^{(1)}$, + rotational invariance:

the only scalar is $\nabla^2\phi = \frac{3}{2}\mathcal{H}^2\Omega_m\delta$

$\Rightarrow \delta_h = b_1\delta$ linear bias

2^{nd} order (quadratic in ϕ) (since $\phi = \phi_v + O(\phi^2)$
$\phi^2 = \phi_v^2 + O(\phi^3)$)

$\rightarrow$ only two indep operators: δ^2 ($\propto (\nabla^2\phi)^2$) and

$$\mathcal{G}_2(\phi) = (\nabla_i\nabla_j\phi)(\nabla_i\nabla_j\phi) - (\nabla^2\phi)^2 \quad \text{Galileon}$$

$$\Rightarrow \delta_h = b_1\delta + \frac{b_2}{2}\delta^2 + b_{\mathcal{G}_2}\mathcal{G}_2(\phi)$$

"Non local" BIAS !!

The generation of "non local" bias can be understood by gravitational dynamics, i.e. by solving the eq. of motion

$$\varphi = \begin{pmatrix} \delta \\ -\frac{\theta}{f_+} \\ \delta_h \\ -\frac{\theta_h}{f_+} \end{pmatrix} e^{-\eta} \qquad \Omega = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -3/2 & 3/2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -3/2 & 0 & 0 & 3/2 \end{pmatrix}$$

$$\Rightarrow (\delta_{ab}\partial_\eta + \Omega_{ab})\varphi_b = e^{\eta}\gamma_{abc}\varphi_b\varphi_c$$

$$\gamma_{343} = \gamma_{121} \qquad \gamma_{444} = \gamma_{222}$$

propagator $\rightarrow$ 4 eigenmodes: growing $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ ~ const

decaying $\begin{pmatrix}1\\-3/2\\1\\-3/2\end{pmatrix} \sim e^{-5/2\,\eta}$

iso-density $(\ldots) \sim e^{-\eta}$

iso-density-velocity $(\ldots) \sim e^{-\frac{3}{2}\eta}$

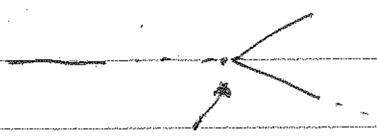
start with $\varphi_a^{in} = \varphi(k) \begin{pmatrix} 1 \\ 1 \\ (b_\delta^*) \\ (b_v^*) \end{pmatrix}$ ← density bias, ← velocity bias } "Lagrangian"

$\boxed{\text{linear theory}}$

$$\rightarrow \varphi_a(k,\eta) = \varphi(k) \begin{pmatrix} 1 \\ 1 \\ 1 - e^{-\eta}(b_\delta^* - 1) - 2(e^{-\eta} + e^{-\frac{3}{2}\eta})(b_v^* - 1) \\ 1 + e^{-\frac{3}{2}\eta}(b_v^* - 1) \end{pmatrix}$$

$q \to \infty$: linear debiasing $q \to (\vdots)$

$\boxed{2^{nd} \text{ order PT}}$



δ^{b}_{lin} $\quad q_{in} \equiv q_x$

halo formation

δ^{ϵ}_{lin}

Momentum dependent vertex $\to \propto (q_1 \cdot q_2)^2$ generates up to ~~dipole~~ quadrupole moments

We are interested in deviation from the local bias:

$$\delta_h^{Nloc} \equiv \delta_h - \delta_h^{Local} = \delta_h - \sum_n \frac{b_n}{n!} \delta^n$$

Consider $\chi \doteq \delta_h - b_1 \delta = \delta_h^{Nloc} + \sum_2^\infty{}_n \frac{b_n}{n!} \delta^n$

and study it at second order PT:

$$\chi^{(2)} = \delta_h^{Nloc\,(2)} + \frac{b_2}{2} \delta_{lin}^2$$

Expand in Legendre polynomials

$$\chi^{(2)}(x) = \int \frac{d^3 q_1\, d^3 q_2}{(2\pi)^3 (2\pi)^3} e^{-i(\vec{q}_1 + \vec{q}_2)\cdot \vec{x}} \delta_{lin}(q_1)\, \delta_{lin}(q_2)$$

$$\times \sum_{\ell=0}^{\infty} P_\ell(\mu)\, \chi_\ell^{(2)}(q_1, q_2) \qquad \mu = \hat{q}_1 \cdot \hat{q}_2$$

The local term, $\frac{b_2}{2} \delta_{lin}^2$, contributes to the monopole.

$\ell = 1, 2$ signal the existence of NON-LOCAL BIAS $(\delta_h^{Nloc} \neq 0)$

(Full result, see Chan, Scoccimarro, Sheth, 2012)

Take $b_v^* = 0$ $\Rightarrow$ $\chi_0^{(2)} = \frac{b_2^*}{2} + \frac{4}{21} e^{\eta}(b_1^* - 1)$

$\eta \to \infty$

$\chi_1^{(2)} = 0$

$\chi_2^{(2)} = -\frac{4}{21} e^{\eta}(b_1^* - 1)$ $\Leftarrow$ NON LOCAL TERM

(If $b_v^* \neq 0$, then also a dipole is generated)

The quadrupolar NON-local term shows that even if bias is LOCAL at the "initial/formation" time, it is generated later on.

It can be expressed in terms of $\mathcal{G}_2(\Phi)$

$$\boxed{\delta_h = b_1 \delta + \frac{b_2}{2} \delta^2 + b_{\mathcal{G}_2} \mathcal{G}_2(\Phi)}$$

tested in numerical simulations

(δ_h not constant in δ-constant surfaces)

See Renormalized Halo Bias by Assassi et al 1402.5916

- Can be generalised at higher orders
- G_2 ... should be considered in measurements of Bispectrum and so on

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Redshift space distortions

Total velocity of a particle at physical coordinate

$$\vec{R} = a\vec{x} \; : \quad \frac{d\vec{R}}{dt} = \underset{\text{Hubble flow}}{\mathcal{H}\vec{x}} + \underset{\text{peculiar velocity}}{\frac{\vec{p}}{am}} \qquad \mathcal{H} = \frac{d\ln a}{d\tau}$$

$$\frac{\vec{p}}{am} = \vec{v} = \frac{d\vec{x}}{d\tau}$$

assuming, incorrectly, that the whole velocity is due to the Hubble flow, we obtain the Redshift space position

$$\vec{s} = \vec{x} + \frac{\vec{p}\cdot\hat{x}}{am\mathcal{H}}\hat{x} \sim \vec{x} + \frac{p_z}{am\mathcal{H}}\hat{z}$$

line of sight

distant observer approx

Microscopic dist. functions: Real space $f(\vec{x},\vec{p},\tau) = \sum_n \delta_D(\vec{x}-\vec{x}_n(\tau))\delta_D(\vec{p}-\vec{p}_n(\tau))$

redshift space $f_s(\vec{s},\vec{p},\tau) = \sum_n \delta_D(\vec{s}-\vec{s}_n(\tau))\delta_D(\vec{p}-\vec{p}_n(\tau))$

RS number density

$$\boxed{n_s(\vec{s},\tau) = \int d^3p \, f_s(\vec{s},p,\tau) = \int d^3\vec{p} \int d^3\vec{x} \, f(\vec{x},p,\tau) \, \delta_D\left(\vec{s}-\vec{x}-\frac{p_z}{a m \mathcal{H}}\hat{z}\right)}$$

$$\int d^3s \, n_s(\vec{s},\tau) = \int d^3\vec{x} \, n(\vec{x},\tau) = n_0 V_{\text{tot}}$$

holds even in presence of multistreaming

$n = n_0(1+\delta)$

$n_s = n_0(1+\delta_s)$

+

FOURIER TRANSF

$$\Rightarrow \delta_D(\vec{k}) + \delta_s(\vec{k},\tau) = \int d^3\vec{s} \, e^{i\vec{k}\cdot\vec{s}} \left(1+\delta_s(\vec{s},\tau)\right)$$

$$= n_0^{-1} \int d^3p \, d^3x \, e^{i\vec{k}\cdot\vec{x}} \, e^{i\frac{k_z p_z}{a m \mathcal{H}}} f(\vec{x},\vec{p},\tau)$$

To proceed, we can expand $e^{i\frac{k_z p_z}{a m \mathcal{H}}}$ and express the RHS in terms of an infinite series in moments:

$$\Rightarrow \delta_s(\vec{k},\tau) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{i k_z}{\mathcal{H}}\right)^{\ell} T_z^{\ell}(\vec{k},\tau)$$

with $T_z^{\ell}(\vec{k},\tau)$ the F.T. of $T_z^{\ell}(\vec{x},\tau) = \frac{1}{n_0} \int d^3p \left(\frac{p_z}{a m}\right)^{\ell} f(\vec{x},\vec{p},\tau)$

$$\rightarrow \quad T_z^0(\vec{x}, \tau) = \delta(\vec{x}, \tau)$$

$$T_z^1 = (1+\delta(\vec{x}))\, v_z(\vec{x})$$

$$T_z^2 = (1+\delta)\left(v_z^2(x) + \sigma_{zz}(x)\right)$$

if now we

1) TRUNCATE TO $\ell = 1$;
2) ASSUME linear theory
3) ASSUME linear bias ($\delta_g = b\,\delta_m$) (and no velocity bias $v_g = v$)

$$\hookrightarrow \quad \frac{i k_z v_z^s}{H} = \frac{k_z^2}{H k^2}\theta_s = -\mu^2\left(-\frac{\theta}{H}\right) = -\mu^2 f \delta = -\mu^2 \frac{f}{b}\delta_g$$

$$\Rightarrow \delta_{g,s}(\vec{k}, \tau) \simeq \delta_g(\vec{k}, \tau)\left(1 - \mu^2\beta\right) \qquad \beta = \frac{f}{b}$$

$$\Rightarrow P_{g,s}(\vec{k}) = (1 - \mu^2\beta)^2 P_g(\vec{k}) \qquad \underline{\text{KAISER FORMULA}}$$

But 1) + 2) + 3) do not hold at small scales

In particular 1).

McDonald, Seljak et al study the momentum correlations

the effects of $\langle \delta(x)\, \delta(y)\, \sigma^2_{zz}(y) \rangle$ or $\langle \delta(x)\, \sigma_{zz}(x)\, \delta(y)\, \sigma_{zz}(y) \rangle$

terms is not negligible even at large scales: UV-IR mixing

(σ is a UV effect, but manifests itself also at large scales).

SPT calculation not very successful (to be expected)

- Resummation of the σ_{zz}
- phenomenological models $e^{-\sigma_{zz} k^2 \mu^2}$ FINGER OF GOD RESUMMATION
- hybrid approaches (measure of σ effects in N-body)

A Lot to be done!